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A Boolean-valued Models Approach to L^0-Convex Analysis, Conditional Risk and Stochastic Control

Una Aproximación al Análisis L^0-Convexo, al Riesgo Condicional y al Control Estocástico Basada en Modelos con Valores Booleanos

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Contents

"We must ask whether there is any interest in these nonstandard models aside from the independence proof; that is, do they have any mathematical interest? The answer must be yes, but we cannot yet give a really good argument."

DANA SCOTT, 1969

Resumen

A lo largo de las dos últimas décadas, la teoría de representación dual de medidas de riesgo, la cual tuvo sus orígenes en el artículo de seminario [5], ha sido un área activa y prolífica de investigación dentro de la matemática financiera cf. [3, 4, 12, 24, 27, 28, 35, 39, 40, 45, 68, 71, 87]. Aztner et al. [5] introdujeron y estudiaron el concepto de medida de riesgo coherente. Föllmer y Schied [39], e independientemente Fritelli y Gianin [45], introdujeron posteriormente el concepto más general de medida de riesgo convexa. Ambos tipos de medidas de riesgo tienen una configuración estática, en la cual hay sólo dos instantes de tiempo relevantes: hoy 0 y mañana T. En este caso, un mercado financiero se modela con un espacio de probabilidad $(\Omega, \mathcal{E}, P)$ donde en $\mathcal{E}$ se halla codificada la información de mercado que estará disponible en el instante T. Un *medida de riesgo* es una función ρ que asigna a cada variable aleatoria x medible en $\mathcal{E}$, la cual modela un pago final, un número real $\rho(x)$ el cual cuantifica el riesgo de x.

Una situación más intrincada es cuando tenemos una configuración dinámica de tiempo, en la cual se tiene en cuenta la llegada de nueva información en un instante intermedio $0 < t < T$. En este caso, la información de mercado disponible en tiempo t se encuentra codificada en una sub-σ-álgebra $\mathcal{F}$ de $\mathcal{E}$. Entonces una *medida de riesgo condicional* es una aplicación ρ_t que asigna a cada pago final, modelado por una variable aleatoria x medible en $\mathcal{E}$, otra variable aleatoria $\rho_t(x)$ medible en $\mathcal{F}$, la cual cuantifica en tiempo t el riesgo que surge de x.

Tal como se explica en [36], si bien el análisis convexo clásico se aplica perfectamente al caso de un solo periodo, la aplicación al caso de periodos múltiples es más bien delicada, requiriéndose a veces de fuertes criterios de selección medible. Por ejemplo, considérense propiedades de una medida de riesgo condicional ρ_t tales como la convexidad, la continuidad, la derivabilidad, etc. Estas propiedades han de cumplirse puntualmente en cada $\omega \in \Omega$, pero esto debería satisfacerse de algún modo dependiendo mediblemente de ω con el fin de permitir un esquema recursivo de múltiples periodos.

Estas dificultades han motivado algunos nuevos desarrollos en análisis funcional. Por ejemplo, Filipovic, Kupper, y Vogelpoth [36] proponen un enfoque basado en módulos, donde los escalares son variables aleatorias en lugar de números reales. Concretamente, consideran módulos sobre $L^0(\mathcal{F})$, el espacio ordenado (de clases de equivalencia) de variables aleatorias medibles en $\mathcal{F}$. Para este fin, establecen el concepto módulo localmente L^0-convexo —un equivalente en módulos de los espacios localmente convexos— y desarrollan una versión aleatorizada del análisis convexo, proporcionando teoremas de separación por hiperplanos y teoremas de representación dual tipo Fenchel-Moreau. Más recientemente, Drapeau, Jamnesahn, Karliczek, y Kupper [32], en un nivel más abstracto, establecen un nuevo marco llamado *teoría de conjuntos condicionales*, el cual proporciona un lenguaje formal junto a métodos constructivos que generalizan las técnicas del análisis L^0-convexo de una manera estructurada. Sobre esta base, encuentran demostraciones para muchas versiones condicionales de resultados del análisis funcional, empleando una formulación que depende de forma consistente de los elementos del álgebra de probabilidad asociada a la σ-álgebra $\mathcal{F}$ que modela la información de mercado. Tal como explican los autores, el número de equivalentes condicionales de teoremas convencionales

proporcionados por la teoría de conjuntos condicionales arroja evidencia de la posibilidad de un discurso matemático basado en la noción de conjunto condicional.

Ahora, vayamos al otro extremo de las matemáticas. Los *modelos con valores Booleanos* son una herramienta de la lógica matemática que fue desarrollada como una manera de formalizar el método de *forzamiento* que Paul Cohen ideó para resolver el primer problema de la famosa lista de Hilbert: es imposible demostrar o refutar que todo conjunto infinito de números reales se encuentra en correspondencia biyectiva o bien con los números naturales o bien con toda la recta real [25]. La teoría fue primeramente formulada por Scott [104] basándose en algunas ideas de Solovay, mientras que Vopěnka creó independientemente una teoría similar. La idea detrás de los modelos con valores Booleanos es construir a partir de un álgebra de Boole completa $(\mathcal{A}, \vee, \wedge, \cdot^c, I, 0)$ una clase $V^{(\mathcal{A})}$, la cual es un modelo de la teoría de conjuntos de Zermelo-Fraenkel con el axioma de elección, abreviado bajo el acrónimo ZFC, de modo que los valores de verdad de las proposiciones no se limitan a "verdadero", digamos I, o "falso", digamos 0, sino que pueden tomar cualquier valor a en $\mathcal{A}$. El llamado *principio de transferencia de los modelos con valores Booleanos* afirma que cualquier teorema conocido de ZFC tiene valor de verdad Booleano I dentro del universo $V^{(\mathcal{A})}$. Esto es, si φ es un teorema de la teoría de conjuntos, la afirmación "φ tiene valor de verdad Booleano I" es también un teorema. Esto proporciona una tecnología para expandir el contenido de resultados conocidos de la teoría de conjuntos a nuevos teoremas, los cuales tienen a menudo un contenido no obvio. En otras palabras, cada teorema disponible que haya sido demostrado por medios convencionales da lugar a una colección de nuevos teoremas enumerados en las diferentes álgebras de Boole completas.

Un ejemplo de aplicación de esta tecnología es el llamado *análisis con valores Booleanos*. Este campo es una rama del análisis funcional, la cual estudia propiedades de objetos matemáticos comparando sus representaciones en dos modelos con valores Booleanos distintos. El análisis con valores Booleanos fue iniciado por Gordon [47] y Takeuti [107], y ha experimentado un fructífero y profundo desarrollo gracias a la excelente investigación de Kusraev y Kutateladze. Para una cuenta completa remitimos al lector a [77, 81] y sus extensivas listas de referencias y para una breve introducción lo remitimos a [82].

Sobre esta base, este trabajo tiene por objetivo:

Primero, establecer una aproximación sistemática al análisis L^0-convexo por medio del análisis con valores Booleanos y poner a nuestra disposición todas las poderosas herramientas de una teoría matemática bien desarrollada y profunda, y mostrar que el análisis L^0-convexo se ajusta de manera natural al marco del análisis con valores Booleanos. A saber, proporcionamos un método que, de forma sistemática, proporciona a partir de cada teorema clásico del análisis convexo un teorema análogo en módulos, sin la necesidad de una demostración paso a paso del mismo, gracias al principio de transferencia de los modelos con valores Booleanos. Esto da como resultado una analogía completa en módulos del análisis convexo clásico, la cual no sólo cubre los teoremas de separación por hiperplanos y los teoremas de representación dual del tipo Fenchel-Moreau establecidos en [36], sino también el equivalente en módulos de cualquier resultado clásico del análisis convexo. Entre muchos otros, se proporcionan versiones aleatorizadas de famosos resultados como el teorema del punto fijo de Brouwer y el teorema de compacidad de James.

Segundo, explorar nuevas aplicaciones a la teoría dual de medidas de riesgo condicionales y a la teoría de control estocástico. Específicamente, establecemos un principio de transferencia desde la teoría dual de medidas de riesgo convencionales a la teoría dual de medidas de riesgo condicionales; esto es, proporcionamos un método para interpretar resultados disponibles sobre la representación dual de medidas de riesgo convencionales como nuevos teoremas sobre la representación dual de medidas de riesgo condicionales. Como ejemplo de aplicación, se establece un teorema general de representación robusta de medidas de riesgo condicionales, el cual cubre aquellos casos en los que una medida de riesgo condicional está definida sobre módulos del tipo L^p, Orlicz y corazones de Orlicz.

Utilizamos también el análisis L^0-convexo para estudiar la existencia de soluciones óptimas de una familia de problemas de optimización de control estocástico en tiempo finito discreto. Se estudian algunos problemas de matemáticas financieras relacionados; por ejemplo, la maximización de utilidad dependiente de la riqueza y reparto de utilidad dependiente de la riqueza.

Este trabajo se organiza en cinco capítulos. A continuación, se explica brevemente su contenido:

Capítulo 1: Un aproximación convencional al análisis L^0-convexo: Este trabajo empieza con el estudio de algunas cuestiones abiertas derivadas de una aproximación convencional al análisis L^0-convexo. Sea $(\Omega, \mathcal{F}, P)$ un espacio de probabilidad y denotemos con L^0 el espacio de variables aleatorias medibles en $\mathcal{F}$ e identificadas cuando coinciden en casi todo $\omega \in \Omega$. Supongamos que E es un L^0-módulo. Dada una partición contable $(A_k)_{k\in\mathbb{N}}$ de Ω en elementos de $\mathcal{F}$ y una sucesión $(x_k)_{k\in\mathbb{N}}$ en E, decimos que un elemento de x de E es una concatenación contable de (x_k) a lo largo de (A_k) si $1_{A_k}x = 1_{A_k}x_k$ para cada $k \in \mathbb{N}$, donde 1_A denota la clase de equivalencia de la función característica definida sobre $A \in \mathcal{F}$. Decimos que un subconjunto S de E es *estable* si la concatenación de cualquier sucesión en S a lo largo de cualquier partición de Ω es un elemento de S. La nociones de concatenación contable y de estabilidad son cruciales en el análisis L^0-convexo (véase eg [36, 51, 23]). Por medio de diversos ejemplos de casos límite de L^0-módulos y resultados, en este capítulo estudiamos cuestiones relacionadas con la estabilidad en L^0-módulos.

Por ejemplo, una cuestión abierta que surge de [36] es si cualquier topología localmente L^0-convexa es inducida por una familia de L^0-seminormas (i.e. seminormas con valores en L^0). En relación a este problema, se demuestra que una topología localmente L^0-convexa es inducida por una familia de L^0-seminormas si y sólo si admite una base de entornos de $0 \in E$ consistente en subconjuntos estables y L^0-convexos (véase Teorema 1.2.1). Además, se proporciona un contraejemplo de una topología localmente L^0-convexa que no es inducida por ninguna familia de L^0-seminormas. Esto pone de relieve que la condición de estabilidad no puede ser suprimida (véase Ejemplo 1.2.1).

Se dice que un L^0-módulo E tiene la *propiedad de concatenación contable* si para cada sucesión (x_k) en E y cualquier partición (A_k) de Ω en elementos de $\mathcal{F}$ existe una única concatenación contable, digamos $\sum_{k\in\mathbb{N}} 1_{A_k}x_k$, de (x_k) a lo largo de (A_k). Demostramos que un L^0-módulo libre tiene la propiedad de concatenación contable si y sólo si tiene rango finito. Se estudian L^0-módulos cuya clase de subconjuntos estables es aditiva. Se demuestra que esta propiedad es estrictamente más débil que la propiedad de concatenación contable; sin embargo, tal como se demuestra, la primera propiedad es suficiente para establecer un equivalente en L^0-módulos del teorema de Mazur clásico (véase Teorema 1.3.1 y Corolario 1.3.1).

Si E tiene la propiedad de concatenación contable, entonces tiene sentido definir la concatenación de una familia contable (S_k) de subconjuntos no vacíos de E a lo largo de una partición (A_k), la cual denotamos por $\sum_{k\in\mathbb{N}} 1_{A_k}S_k$. Decimos que una colección $\mathscr{U}$ de subconjuntos de E es estable, si las concatenaciones contables de los miembros de $\mathscr{U}$ son de nuevo miembros de $\mathscr{U}$. Un *módulo localmente estable L^0-convexo* es un módulo localmente L^0-convexo con la propiedad de concatenación contable que admite una base de entornos de $0 \in E$ consistente en subconjuntos estables L^0-convexos de modo que ella misma es estable. Estas propiedades de estabilidad son cruciales para probar los teoremas de separación y dualidad de [36]. Caracterizamos las topologías localmente estable L^0-convexas en términos de familias de L^0-seminormas (véase Teorema 1.4.1). Se proporcionan ejemplos de topologías localmente L^0-convexas que no cumplen estas condiciones. Por ejemplo, demostramos que las importantes topologías débil y débil-$*$ no son, en general, localmente estables L^0-convexas (véase Ejemplo 1.4.2). Esto motiva la definición de las versiones estables de las topologías débil y débil-$*$.

Una revisión de la literatura existente sobre el análisis L^0-convexo muestra que apenas hay resultados que usen la compacidad clásica, y los pocos que se pueden encontrar son en dirección negativa. Por ejemplo, en [50] se demuestra que no es posible establecer un análogo del teorema de Banach-Alaoglu para L^0-módulos cuando los dotamos de la topología de la convergencia estocástica. Por tanto, una pregunta natural es si es posible establecer un enfoque basado en L^0-módulos de la compacidad convencional. En este sentido, en este capítulo demostramos que, excepto en casos límite, los módulos localmente L^0-convexo son anticompactos; es decir, los único subconjuntos compactos son finitos (véase Teorema 1.4.2). Esto demuestra que la clase de subconjuntos compactos de un módulo localmente L^0-convexo es tan pequeña que hace imposible establecer cualquier resultado significativo de compacidad. Esto motiva el uso de formas más débiles de compacidad, que serán explotadas más adelante en los próximos capítulos.

Capítulo 2: Una aproximación al análisis L^0-convexo basada en modelos con valores Booleanos: En este capítulo, desarrollamos un principio de transferencia del análisis convexo clásico al análisis L^0-convexo basándonos en los modelos con valores Booleanos. Supongamos que los eventos del mercado en una fecha futura $t > 0$ son modelados por el espacio de probabilidad $(\Omega, \mathcal{F}, P)$. Consideremos el álgebra de probabilidad $(\bar{\mathcal{F}}, \bar{P})$ donde $\bar{\mathcal{F}}$ se define identificando eventos cuya diferencia simétrica tiene probabilidad 0 y $\bar{P}(a) := P(A)$, donde $a \in \bar{\mathcal{F}}$ es representado por $A \in \mathcal{F}$. Entonces $\mathcal{A} := \bar{\mathcal{F}}$ tiene estructura de álgebra de Boole completa, y es posible construir el modelo con valores Booleanos $V^{(\mathcal{A})}$ asociado a $\mathcal{A}$, así como un lenguaje de primer orden $\mathcal{L}^{(\mathcal{A})}$ el cual permite un razonamiento completamente análogo al de la teoría de conjuntos, pero basado en los miembros de la clase $V^{(\mathcal{A})}$. Cada fórmula φ del lenguaje $\mathcal{L}^{(\mathcal{A})}$ toma un valor de verdad Booleano, digamos $[\![\varphi]\!]$, el cual es un elemento de $\mathcal{A}$. Decimos que un enunciado φ es válido en $V^{(\mathcal{A})}$ si ocurre que $[\![\varphi]\!] = I$. Entonces, resulta que $V^{(\mathcal{A})}$ junto con el valor de verdad Booleano es un modelo de la teoría de conjuntos basado en el sistema de axiomas de Zermelo-Fraenkel junto con el axioma de elección (ZFC). En particular, cualquier teorema disponible de la teoría de conjuntos tiene un valor de verdad Booleano I. En otras palabras, si φ es un teorema, entonces "φ tiene un valor de verdad Booleano I" es nuevamente un teorema. Este notable hecho es conocido como *principio de transferencia de modelos con valores Booleanos*.

En este capítulo, se demuestra que cualquier módulo localmente estable L^0-convexo E se puede convertir en un espacio localmente convexo clásico $E\!\uparrow$ dentro del modelo con valores booleanos $V^{(\mathcal{A})}$. Debido al principio de transferencia, $E\!\uparrow$ satisface dentro de $V^{(\mathcal{A})}$ cada uno de los teoremas conocidos sobre espacios localmente convexos. Por lo tanto, si logramos interpretar un teorema sobre espacios localmente convexos como un enunciado sobre módulos localmente L^0-convexos, habremos probado de hecho un nuevo teorema sobre módulos localmente L^0-convexos. Sobre esta base, revisamos varias nociones que provienen del análisis L^0-convexo, y mostramos que todas ellas tienen un significado convencional dentro de $V^{(\mathcal{A})}$. Esto nos proporciona una gran cantidad de elementos los cuales se pueden ensamblar para construir diferentes teoremas en L^0-módulos análogos a teoremas del análisis convexo clásico.

Se proporcionan diversos ejemplos de aplicación. En particular, mostramos que los resultados principales de [36] se siguen automáticamente del principio de transferencia; las versiones en L^0-módulos del teorema de separación de Hahn-Banach, del teorema de la dualidad de Fenchel, y del teorema de Fenchel-Rockafeller demostrados en [36] son reformulaciones de sus versiones clásicas dentro de $V^{(\mathcal{A})}$. Además, haciendo uso del principio de transferencia, establecemos teoremas en L^0-módulos análogos a teoremas clásicos tales como los famosos teoremas del punto fijo de Brouwer (véase Teorema 2.4.1), de la categoría Baire (véase Teorema 2.4.6), de Eberlein-Šmulian (véase Teorema 2.4.8), de Krein-Šmulian (véase Teorema 2.4.12), de Mazur (véase Teorema 2.4.13) y de compacidad de James (ver Teorema 2.4.9). Asimismo, se establece una versión perturbada de este último (ver Teorema 2.4.10). Sin embargo, el número de ejemplos de aplicación puede aumentarse fácilmente.

Capítulo 3: Una aproximación a la teoría de conjuntos condicionales basada en modelos con valores Booleanos: La teoría de conjuntos condicionales fue introducida en [32]. En este marco se extiende la noción de concatenación contable en L^0-módulos a estructuras más generales. Específicamente, dada un álgebra de Boole completa $\mathcal{A}$, un conjunto condicional de un conjunto no vacío X es un conjunto $\mathbf{X}$ para el cual existe una aplicación sobreyectiva $(x,a) \mapsto x|a$ de $X \times \mathcal{A}$ a $\mathbf{X}$ tal que:

- $x|a = y|b$ implica $a = b$;
- $x|b = y|b$ y $a \leq b$ implica $x|a = y|a$;
- para cualquier partición de la unidad $(a_i)_{i\in\Delta}$ en $\mathcal{A}$ y cada familia $(a_i)_{i\in\Delta}$ de elementos de X existe exactamente un elemento $x \in X$ tal que $x|a_i = x_i|a_i$ para todo $i \in I$.

Basándose en la noción de conjunto condicional, en la teoría de conjuntos condicionales se da sentido a versiones condicionales de diferentes objetos matemáticos (funciones condicionales, relaciones binarias condicionales, topologías condicionadas, etc.) y además se formulan versiones condicionales de resultados clásicos, las cuales son probadas adaptando demostraciones clásicas.

En este capítulo, demostramos que la teoría de conjuntos condicionales puede verse como una instancia particular de los modelos con valores Booleanos. A saber, consideramos el modelo con valores Booleanos $V^{(\mathcal{A})}$ asociado a $\mathcal{A}$ y probamos que cualquier conjunto condicional se puede convertir en un conjunto $\mathbf{X}\uparrow$ dentro de $V^{(\mathcal{A})}$; además, mostramos que esta asignación es una equivalencia entre la categoría de conjuntos condicionales y funciones condicionales, y la categoría de conjuntos no vacíos y funciones dentro de $V^{(\mathcal{A})}$. Revisamos los diferentes elementos de la teoría de conjuntos condicionales y estudiamos su interpretación dentro de $V^{(\mathcal{A})}$: mostramos que las relaciones binarias condicionales, las topologías condicionales, los espacios métricos condicionales, etc., pueden interpretarse respectivamente como relaciones binarias, topologías, espacios métricos, etc. dentro de $V^{(\mathcal{A})}$. Todo esto muestra que la teoría de conjuntos condicionales es una interpretación en modelos con valores Booleanos de la teoría de conjuntos clásica. Entonces, como aplicación del principio de transferencia de los modelos con valores Booleanos, las versiones condicionales de teoremas clásicos son ciertas sin la necesidad de una demostración paso a paso por medios convencionales.

Como ejemplos de aplicación de este método, proporcionamos fácilmente versiones condicionales del teorema del punto fijo de Brouwer, del teorema de la aplicación abierta, del teorema de la categoría Baire, del principio de acotación uniforme, del teorema de Eberlein-Šmulian, del teorema de Amir-Lindenstrauss y versiones del teorema de compacidad de James.

Asimismo, proporcionamos pruebas por medios convencionales de algunos de estos teoremas mediante las técnicas de los conjuntos condicionales, las cuales se pueden encontrar en el Apéndice al final de este trabajo junto con un estudio exhaustivo de las topologías débiles condicionales (véase Apéndice B)).

Capítulo 4: Representación robusta de medidas de riesgo condicionales: En este capítulo, establecemos un principio de transferencia de la teoría de dualidad de medidas de riesgo convencionales a la teoría de dualidad de medidas de riesgo condicional; es decir, desarrollamos un método para reinterpretar teoremas de representación dual de medidas de riesgo clásicas como nuevos teoremas de representación dual de medidas de riesgo condicionales.

Pongamos este problema en contexto. Supongamos que la información de mercado es modelada por el espacio de probabilidad $(\Omega, \mathcal{E}, P)$, y consideremos un subespacio sólido $\mathcal{X}$ de $L^0(\mathcal{E})$ el cual contiene las funciones constantes. El espacio $\mathcal{X}$ modela todos los posibles pagos finales. El riesgo de cualquier $x \in \mathcal{X}$ es cuantificado por una función $\rho : \mathcal{X} \to \mathbb{R}$ que satisface:

1. *Convexidad:* $\rho(rx + (1-r)y) \leq r\rho(x) + (1-r)\rho(y)$ para todo $r \in \mathbb{R}$ con $0 \leq r \leq 1$;
2. *Monotonía:* Si $x \leq y$, entonces $\rho(y) \leq \rho(y)$;
3. *Invariancia monetaría:* Si $r \in \mathbb{R}$, entonces $\rho(x + r) = \rho(x) - r$.

Tal función es llamada *medida de riesgo convexa.*

El espacio dual de Köthe de $\mathcal{X}$ se define como

$$\mathcal{X}^x := \{y \in L^0(\mathcal{E}) \colon E_P[|xy|] < \infty, \, \forall x \in \mathcal{X}\}.$$

Clásicamente, la teoría dual de medidas de riesgo estudia cuándo una medida de riesgo convexa ρ admite una representación

$$\rho(x) = \sup\{E_P[xy] - \alpha(y) \colon y \in \mathcal{X}^x\} \quad \text{para todo } x \in \mathcal{X},$$

para alguna función de penalización $\alpha : \mathcal{X} \to \mathbb{R} \cup \{\infty\}$. También, estudia cuándo la representación dual de arriba se alcanza, esto es,

$$\rho(x) = \max\{E_P[xy] - \alpha(y) \colon y \in \mathcal{X}^x\} \quad \text{para todo } x \in \mathcal{X}.$$

Por ejemplo, para medidas de riesgo convexas definidas en el espacio L^∞ de variables aleatorias acotadas, la representación con supremo es equivalente a la semi-continuidad inferior en orden de ρ, también llamada propiedad de Fatou (véase [28, 40]). En este caso, Jouini, Schachermayer y Touzi [68] demostraron que convertir el supremo en un máximo es equivalente a la continuidad en orden (propiedad de Lebesgue) de ρ, lo cual es también equivalente la compacidad débil de los conjuntos de subnivel de la función de penalización α.

Supongamos ahora que $\mathcal{F} \subset \mathcal{E}$ es una sub-σ-álgebra en la cual se halla codificada la información de mercado disponible en una fecha intermedia $0 < t < T$. Ahora, los pagos finales van a ser modelados por un $L^0(\mathcal{F})$-submódulo $\mathscr{X}$ de $L^0(\mathcal{E})$ estable y sólido el cual contiene $L^0(\mathcal{F})$. Entonces, una medida de riesgo condicional es una función $\rho : \mathscr{X} \to L^0(\mathcal{F})$ satisfaciendo para todo $x, y \in \mathscr{X}$:

1. $L^0(\mathcal{F})$*-convexidad:* $\rho(\eta x + (1-\eta)y) \leq r\rho(x) + (1-r)\rho(y)$ para todo $\eta \in L^0(\mathcal{F})$ con $0 \leq \eta \leq 1$;
2. *Monotonía:* $x \leq y$ implica $\rho(y) \leq \rho(x)$;
3. *Invarianza* $L^0(\mathcal{F})$*-monetaria:* $\rho(x + \eta) = \rho(x) - \eta$ para todo $\eta \in L^0(\mathcal{F})$.

Este tipo de medidas de riesgo fueron introducidas por Filipovic et al. [37].

Definimos el $L^0(\mathcal{F})$-módulo dual de Köthe de $\mathscr{X}$ como

$$\mathscr{X}^x := \{y \in L^0(\mathcal{E}) \colon E_P[|xy||\mathcal{F}] < \infty, \, \forall x \in \mathscr{X}\},$$

donde $E_P[\cdot|\mathcal{F}]$ denota la esperanza condicional extendida.

Sea $\mathcal{A} := \overline{\mathcal{F}}$ el álgebra de probabilidad asociada a $\mathcal{F}$ y considérese el correspondiente modelo con valores Booleanos $V^{(\mathcal{A})}$. En este capítulo, demostramos que $L^0(\mathcal{E})$ puede ser intenterpretado como un espacio $L^0(\mathcal{E})\!\uparrow$ de variables aleatorias dentro del modelo $V^{(\mathcal{A})}$; cualquier $L^0(\mathcal{F})$-submódulo $\mathscr{X} \subset L^0(\mathcal{E})$ puede ser interpretado como un subespacio sólido $\mathscr{X}\!\uparrow \subset L^0(\mathcal{E})\!\uparrow$ dentro de $V^{(\mathcal{A})}$; y, más aún, cualquier medida de riesgo condicional $\rho : \mathscr{X} \to L^0(\mathcal{F})$ puede ser vista como una medida de riesgo convexa (de un solo periodo) $\rho\!\uparrow : \mathscr{X}\!\uparrow \to \mathbb{R}$ dentro de $V^{(\mathcal{A})}$.

Debido al principio de transferencia de $V^{(\mathcal{A})}$, $\rho\!\uparrow$ satisface cualquier teorema sobre representación dual de medidas de riesgo convexas; por tanto, cualquiera de esos teoremas puede ser reinterpretado como un enunciado el cual es cumplido por ρ. Como consecuencia, tenemos que cualquier teorema sobre representación dual de medidas de riesgo convexas da lugar a un nuevo teorema de representación de medidas de riesgo condicionales. Esto equivale a un principio de transferencia desde la teoría de dualidad de medidas de riesgo convexas a la teoría de dualidad de medidas de riesgo condicionales.

Por medio de este método, se establece un teorema general de representación robusta el cual caracteriza cuándo una medida de riesgo condicional $\rho : \mathscr{X} \to L^0(\mathcal{F})$ tiene una representación dual

$$\rho(x) = \sup\{E_P[xy|\mathcal{F}] - \alpha(y) \colon y \in \mathscr{X}^x\} \quad \text{for all } x \in \mathscr{X},$$

(véase teorema 4.3.1) y caracteriza, en términos de una condición de compacidad adecuada, cuándo la representación de arriba es alcanzada (véase Theorem 4.3.2), a saber,

$$\rho(x) = \max\{E_P[xy|\mathcal{F}] - \alpha(y) \colon y \in \mathscr{X}^x\} \quad \text{for all } x \in \mathscr{X}.$$

Filipovic et al. [37] propusieron los llamados módulos de tipo L^p como espacios modelo para las medidas de riesgo condicionales. A saber, dado $1 \leq p \leq \infty$, el correspondiente módulo tipo L^p se define como

$$L^p_{\mathcal{F}}(\mathcal{E}) := \{x \in L^0(\mathcal{E}) \colon E_P[|x|^p|\mathcal{F}] < \infty\}.$$

Demostramos que $L^p_{\mathcal{F}}(\mathcal{E})$ puede ser de hecho interpretado como un espacio L^p clásico dentro del modelo $V^{(\mathcal{A})}$. Más aún, mostramos que lo mismo puede ser hecho para módulos análogos a los espacios clásicos de Orlicz y corazones Orlicz. Para cualquiera de estos casos, establecemos un teorema de representación robusta de medidas de riesgo condicionales particular (véanse Teoremas 4.3.3, 4.3.4, 4.3.5, 4.3.6, 4.3.7 y 4.3.8).

Capítulo 5: Aplicaciones a la optimización de control estocástico: En este capítulo se investigan aplicaciones del análisis L^0-convexo a la optimización estocástica de problemas dependientes de parámetros y en tiempo finito discreto.

A continuación, presentamos el problema matemático y esbozamos la estrategia para su solución. Supongamos que tenemos un espacio de probabilidad filtrado $(\Omega, \mathcal{F}, (\mathcal{F})_{t=0}^T, P)$. Consideremos un *generador progresivo* $(v_t)_{t=0}^{T-1}$ que dirige un proceso estocástico adaptado

$$x_{t+1} = v_t(x_t, z_t)$$

con valores en $\mathbb{R}^d$, cuya dinámica depende de un proceso adaptado (z_t) con valores e $\mathbb{R}^d$ modelando las decisiones elegidas recursivamente en un conjunto de control $\Theta_t(x_t)$ dependiente del estado x_t, para cada $t = 0, \ldots, T-1$, satisfaciendo condiciones de medibilidad adecuadas. Dado un *generador regresivo* $(u_t)_{t=0}^T$, el objetivo es maximizar

$$u_0(x_0, \cdot, z_0) \circ \ldots \circ u_{T-1}(x_{T-1}, \cdot, z_{T-1}) \circ u_T(x_T) \tag{1}$$

sobre todos los posibles procesos progresivos $(x_t)_{t=0}^T$ inicializados en un punto inicial x_0, donde $\circ$ denota la composición de las funciones. Según el *principio de Bellman*, el problema global de optimización estocástica (1) se resuelve de forma recursiva (hacia atrás en el tiempo) si todos los problemas locales de un solo período

$$\begin{aligned} y_t(x_t) =& \sup_{z_t \in \Theta_t(x_t)} u_t(x_t, y_{t+1}(v_t(x_t, z_t)), z_t), \quad t = 0, \ldots, T-1 \\ y_T(x_T) =& u_T(x_T) \end{aligned}$$

alcanzan sus máximos. Cada problema local en el tiempo t corresponde a la optimización de una función

$$y_t : L^0(\mathcal{F}_t, \mathbb{R}^d) \longrightarrow L^0(\mathcal{F}_t, \mathbb{R} \cup \{\pm\infty\}).$$

Para encontrar su extremo, utilizamos el equivalente en módulos del hecho elemental de que un funcional real superiormente semi-continua sobre un conjunto compacto alcanza su extremo. Específicamente, dado que trabajamos en dimensión finita, es suficiente que cada función y_t sea inferiormente semi-continua en un conjunto acotado en la $L^0(\mathcal{F}_t)$-norma de $L^0(\mathcal{F}_t, \mathbb{R}^d)$.

Se proporcionan dos teoremas de existencia de soluciones óptimas. A saber, en el Teorema 5.2.1, se asumen condiciones de compacidad los conjuntos de control $\Theta_t(x_t)$. En el Teorema 5.2.3, no se consideran condiciones en el conjunto de control, pero se asumen condiciones más fuertes en los generadores progresivos y regresivos.

Clásicamente, la herramienta utilizada para estudiar optimización de control estocástico óptimo son los llamados *integrandos normales*, véase, por ejemplo, [13, 91, 92, 94, 95, 96, 97] y para una introducción, remitimos a [98]. En este capítulo, demostramos que existe una correspondencia biyectiva entre integrandos normales de $\Omega \times \mathbb{R}^d$ a $\mathbb{R} \cup \{\pm\infty\}$ propios y funciones de $L^0(\mathcal{F}_t, \mathbb{R}^d)$ a $L^0(\mathcal{F}_t, \mathbb{R} \cup \{\pm\infty\})$ propias y sucesionalmente inferiormente semi-continuas y satisfaciendo la propiedad local.

Estudiamos algunos problemas financieros relacionados: Encontramos soluciones óptimas para un problema de maximización de utilidad dependiente de la riqueza bajo restricciones de venta en corto y estrategias de autofinanciadas; estudiamos problemas de maximización de utilidad dependiente de la riqueza para funciones de utilidad entrópicas generalizadas; y encontramos una solución explícita para un problema de reparto de utilidad con funciones de utilidades entrópicas.

Finalmente, finalizamos este capítulo con una nueva demostración basada en el análisis L^0-convexo para el teorema de Dalang-Morton-Willinger, que es la versión en tiempo finito del célebre Primer Teorema Fundamental de Asignación de Precios (see [30]).

Abstract

Over the past two decades the theory of dual representation of risk measures, which was originated in the seminal paper [5], has been an active and prolific area of research within mathematical finance cf. [3, 4, 12, 24, 27, 28, 35, 39, 40, 45, 68, 71, 87]. Aztner et al. [5] introduced and studied the concept of coherent risk measure. Föllmer and Schied [39] and, independently, Fritelli and Gianin [45] later introduced the more general notion of convex risk measure. Both kinds of risk measures are defined in a static setup, in which only two instants of time matter: today 0 and tomorrow T. In this case, a financial market is modelled by a probability space $(\Omega, \mathcal{E}, P)$ where $\mathcal{E}$ encodes the market information that will be available at time T. A *risk measure* is a function ρ that assigns to any $\mathcal{E}$-measurable random variable x, which models a final payoff, a real number $\rho(x)$ which quantifies the riskiness of x.

A more intricate situation is when we have a dynamic configuration of time, in which the arrival of new information at an intermediate date $0 < t < T$ is taken into account. In this case, the information available at t is encoded in a sub-σ-algebra $\mathcal{F}$ of $\mathcal{E}$. Then a *conditional risk measure* is a map ρ_t that assigns to any final payoff, modelled by an $\mathcal{E}$-measurable random variable x, an $\mathcal{F}$-measurable random variable $\rho_t(x)$ which quantifies at time t the risk arisen from x.

As explained in [36], whereas classical convex analysis perfectly applies to the one-period case, its application to the multi-period case seems to be rather delicate, requiring sometimes of heavy measurable selection criteria. For instance, consider the properties of the conditional risk measure ρ_t such as convexity, continuity, differentiability and so on. These properties have to be satisfied ω wise (with $\omega \in \Omega$), but this should somehow be fulfilled measurably depending on ω in order to enable a recursive multi-period scheme.

These difficulties have motivated some new developments in functional analysis. For instance, Filipovic, Kupper, and Vogelpoth [36] proposed a module-based approach, where scalars are random variables instead of real numbers. Namely, they considered modules over $L^0(\mathcal{F})$, the ordered space (of equivalence classes) of $\mathcal{F}$-measurable random variables. For this purpose, they established the concept of locally L^0-convex module —a module analogue of locally convex spaces— and developed a randomized version of convex analysis providing hyperplane separation theorems and dual representation theorems of Fenchel-Moreau type. More recently, Drapeau, Jamnesahn, Karliczek, and Kupper [32], in a more abstract level, established a new framework called *conditional set theory*, which provides a formal language together with constructive methods that generalize the techniques of L^0-convex analysis in a structured way. On this basis, they found proofs for several conditional versions of functional analysis results, employing a formulation that consistently depends on the elements of the probability algebra associated to a σ-algebra $\mathcal{F}$ that models the market information. As explained by the authors, the number of conditional analogues of conventional theorems provided by conditional set theory gives evidence of a full mathematical discourse based on the notion of conditional set.

Now, let us move to the other edge of mathematics. Boolean-valued models are a tool in

mathematical logic that was developed as a way to formalize the method of forcing that Paul Cohen created to solve the first problem in the famous Hilbert's list: it is impossible neither to prove nor to disprove that every infinite set of reals can be bijected either with the natural numbers of with the whole real line [25]. The theory was first formulated by Scott [104] based on some ideas of Solovay, while Vopěnka created independently a similar theory. The idea behind Boolean-valued models is to construct from a given complete Boolean algebra $(\mathcal{A}, \vee, \wedge, \cdot^c, I, 0)$ a class $V^{(\mathcal{A})}$ which is a model for the Zermelo-Fraenkel set theory with the axiom of choice, ZFC for short, so that the truth values of propositions are not limited to "true", say I, and "false", say 0, but instead they can take any value a in $\mathcal{A}$. The so-called *transfer principle of Boolean-valued models* asserts that any known theorem of ZFC has Boolean truth value I within the universe $V^{(\mathcal{A})}$. That is, if φ is a theorem of set theory, the assertion "φ has Boolean truth value I" is also a theorem. This provides a technology for expanding the content of known results of set theory to new theorems which have often non-obvious content. In other words, each available theorem proven by conventional means gives rise to a collection of new theorems enumerated by the different complete Boolean-algebras.

An example of the application of this technology is the so-called *Boolean-valued analysis*. This field is a branch of functional analysis, which studies the properties of a mathematical object by means of comparison between its representations in two different Boolean-valued universes. Boolean-valued analysis was started by Gordon [47] and Takeuti [107] and has undergone a fruitful and deep development due to Kusraev and Kutateladze. For a thorough account we refer the reader to [77, 81] and their extensive lists of references and for a brief introduction we refer to [82].

On this basis, this work aims:

First, to establish a systematic approach to L^0-convex analysis by means of Boolean-valued models and to get at our disposal all the powerful tools of a well developed and deep mathematical theory, and show that L^0-convex analysis naturally fits the framework of Boolean-valued analysis. Namely, we provide a method that, in a systematic way, provides an analogous theorem in modules from each classical theorem of convex analysis, without the necessity of a step-by-step proof of it, thanks to the principle of transfer of Boolean-valued models. This results in a full module analogue of convex analysis which covers not only the hyperplane separation theorems and dual representation theorems of Fenchel-Moreau type established in [36], but also module analogues of any classical convex analysis theorem. Among many others, we provide randomized versions of celebrated theorems such as the Brouwer's fixed point theorem and James' compactness theorem.

Second, to explore new applications to duality theory of conditional risk measures and stochastic control theory. Specifically, we establish a Boolean-valued transfer principle from duality theory of conventional risk measures to duality theory of conditional risk measures; that is, we provide a method to interpret available results on dual representation of conventional risk measures as new theorems on dual representation of conditional risk measures. As an instance of application, we provide a general robust representation theorem for conditional risk measures which covers those cases in which a conditional risk measure is defined on L^p, Orlicz, and Orlicz-heart type modules. We also use L^0-convex analysis to study the existence of optimal solutions of a family of control stochastic problems in finite discrete time. Some related examples of application to problems of financial mathematics are studied; for instance, wealth-dependent utility maximization and wealth-dependent utility sharing.

This work is organized in 5 chapters. Next, we will briefly explain their contents:

Chapter 1: A conventional approach to L^0-convex analysis: This work starts with the study of some open questions derived from a conventional approach to L^0-convex analysis. Let $(\Omega, \mathcal{F}, P)$ be a probability space and denote by L^0 the space of $\mathcal{F}$-measurable random variables identified modulo almost everywhere identity. Suppose that E is an L^0-module. Given a countable partition $(A_k)_{k\in\mathbb{N}}$ of Ω to $\mathcal{F}$ and a sequence $(x_k)_{k\in\mathbb{N}}$ in E, an element x of E is said to be a *countable concatenation* of (x_k) along (A_k) if $1_{A_k}x = 1_{A_k}x_k$ for each $k \in \mathbb{N}$, where 1_A denotes the class of equivalence of the characteristic function defined on $A \in \mathcal{F}$. A subset S of E is said to be *stable* if the concatenation of any sequence in S along any countable partition of Ω is a member of S. The notions of countable concatenation and stability are crucial in L^0-convex analysis (see eg [36, 51]). However, some open questions remain open. By means of several examples of boundary L^0-modules and results, in this chapter we study some topics related to stability in L^0-modules.

For instance, an open question that arises from [36] is whether any locally L^0-convex topology is induced by a family of L^0-seminorms (i.e. seminorms that take values in L^0). Concerning this problem, we prove that a locally L^0-convex topology is induced by a family of L^0-seminorms if and only if it admits a neighborhood base of $0 \in E$ consisting of stable L^0-convex subsets (see Theorem 1.2.1). In addition, we provide a counter-example of a locally L^0-convex topology which is not induced by any family of L^0-seminorms, which exhibits that the condition of stability cannot be dropped (see Example 1.2.1).

An L^0-module E is said to have *the countable concatenation property* if for any sequence (x_k) in E and any countable partition (A_k) of Ω to $\mathcal{F}$ there exists a unique countable concatenation, say $\sum_{k\in\mathbb{N}} 1_{A_k}x_k$, of (x_k) along (A_k). We prove that a free L^0-module has the countable concatenation property if and only if it has finite rank. We study L^0-modules whose class of stable subsets is additive. We show that this property is strictly weaker than the countable concatenation property; however, as proven, the former property is sufficient to establish an L^0-module analogue of the classical Mazur's Theorem (see Theorem 1.3.1 and Corollary 1.3.1).

If E has the countable concatenation property, it makes sense to define the concatenation of a countable family (S_k) of non-empty subsets of E along a countable partition (A_k), which is denoted by $\sum_{k\in\mathbb{N}} 1_{A_k}S_k$. We say that a collection $\mathscr{U}$ of subsets of E is stable, if the countable concatenations of members of $\mathscr{U}$ is again a member of $\mathscr{U}$. A *locally stable L^0-convex module* is a locally L^0-convex module with the countable concatenation property which admits a neighborhood $\mathscr{U}$ base $0 \in E$ which consists of stable L^0-convex subsets and is itself stable. These stability properties are crucial to prove the separation and duality theorems in [36]. We characterize locally stable L^0-modules in terms of inducing families of L^0-seminorms (see Theorem 1.4.1). We provide examples of locally L^0-convex topologies which fail this condition. For instance, we prove that the important module weak and weak-$*$ topologies are not in general locally stable L^0-convex (see Example 1.4.2). This motivates the definition of the stable versions of the weak and weak-$*$ topologies.

A review of the existing literature on L^0-convex analysis shows that there are hardly any results concerning classical compactness, and the few that can be found are in a negative direction. For instance, [50] shows that it is not possible to establish a Banach-Alaoglu theorem for L^0-modules endowed with the topology of stochastic convergence. Therefore, a natural question is whether a module-based approach to conventional compactness can be established. In this regard, in this chapter, we prove that, except boundary cases, a locally L^0-convex module is anti-compact; that is, every compact subset is finite (see Theorem 1.4.2). This exhibits that the class of compact subsets of a locally L^0-convex module is so narrow that makes impossible to establish any meaningful compactness result. This motivates to use weaker forms of compactness, which will be exploited later in the next chapters.

Chapter 2: A Boolean-valued approach to L^0-convex analysis: In this chapter, we develop a Boolean-valued transfer principle from classical convex analysis to L^0-convex analysis. Suppose that the market events at a future date $t > 0$ are modelled by the probability space $(\Omega, \mathcal{F}, P)$. We can consider the probability algebra $(\bar{\mathcal{F}}, \bar{P})$ where $\bar{\mathcal{F}}$ is defined by identifying events whose symmetric difference has probability 0 and $\bar{P}(a) := P(A)$ where $a \in \bar{\mathcal{F}}$ is represented by $A \in \mathcal{F}$. Then $\mathcal{A} := \bar{\mathcal{F}}$ has structure of complete Boolean-algebra, and one can construct the Boolean-valued model $V^{(\mathcal{A})}$ associated to $\mathcal{A}$ and the first-order Boolean-valued language $\mathcal{L}^{(\mathcal{A})}$ which enables a full set-theoretic reasoning on the members of the class $V^{(\mathcal{A})}$. Each formula φ in the language $\mathcal{L}^{(\mathcal{A})}$ takes a Boolean truth value, say $[\![\varphi]\!]$, which is an element of $\mathcal{A}$. We say that a statement φ is valid in $V^{(\mathcal{A})}$ whenever $[\![\varphi]\!] = I$. Then, it turns out that $V^{(\mathcal{A})}$ together with the Boolean truth value is a model of the set theory based on the system of axioms of Zermelo-Fraenkel together with the axiom of choice (ZFC). In particular, any available theorem of set theory has Boolean truth value I. In other words, if φ is a theorem, then "φ has Boolean truth value I" is again a theorem. This remarkable fact is known as the *transfer principle of Boolean-valued models.*

In this chapter, we show that any locally stable L^0-convex module E can be made into a classical locally convex space $E{\uparrow}$ within the Boolean-valued model $V^{(\mathcal{A})}$. Due to the Boolean-valued transfer principle, $E{\uparrow}$ satisfies within $V^{(\mathcal{A})}$ every single available theorem on locally convex spaces. Therefore, if we manage to interpret a known theorem on locally convex spaces as a statement on locally L^0-convex modules, we will have proven in fact a new theorem on locally L^0-convex modules. On this basis, we review several notions that come from L^0-convex analysis, showing that all of them have a conventional meaning within $V^{(\mathcal{A})}$. This provides us with a pull of elements that can be assembled to build up different module analogues of theorems of classical convex analysis.

We provide several instances of application. In particular, we show that the main results in [36] follow automatically by means of the transfer principle; the module analogues of the Hahn-Banach separation theorem, Fenchel duality theorem, or Fenchel-Rockafeller theorem established in [36] are reformulations of their classical versions within $V^{(\mathcal{A})}$. In addition, by using the transfer principle, we establish module analogues of celebrated theorems such as the Brouwer's fixed point theorem (see Theorem 2.4.1), Baire category theorem (see Theorem 2.4.6), Eberlein-Šmulian theorem (see Theorem 2.4.8), Krein-Šmulian theorem (see Theorem 2.4.12), Mazur's theorem (see Theorem 2.4.13), and the James' compactness theorem (see Theorem 2.4.9). In addition, a perturbed version of the latter is provided (see Theorem 2.4.10). However, the number of instances of application can be easily increased.

Chapter 3: A Boolean-valued approach to conditional set theory: Conditional set theory was introduced in [32]. In this framework the notion of countable concatenation in L^0-modules is extended to more general structures. Specifically, given an complete Boolean algebra $\mathcal{A}$, a *conditional set* of a non-empty set X is a set $\mathbf{X}$ such that there is a surjection $(x, a) \mapsto x|a$ from $X \times \mathcal{A}$ to $\mathbf{X}$ such that

- $x|a = y|b$ implies $a = b$;
- $x|b = y|b$ and $a \leq b$ implies $x|a = y|a$;
- for any partition of the unity $(a_i)_{i \in \Delta}$ in $\mathcal{A}$ and each family $(a_i)_{i \in \Delta}$ of elements of X there exists exactly one element $x \in X$ such that $x|a_i = x_i|a_i$ for all $i \in \Delta$.

Based on the notion of conditional set, in conditional set theory it is given meaning to conditional versions of different mathematical objects (conditional functions, conditional binary relations, conditional topologies and so on) and also conditional versions of classical results are formulated and proven by adapting existing classical proofs.

In this chapter, we prove that conditional set theory can be seen as a particular instance of Boolean-valued models. Namely, we consider the Boolean-valued model $V^{(\mathcal{A})}$ associated to $\mathcal{A}$

and prove that any conditional set can be made into a set $\mathbf{X}\uparrow$ within $V^{(\mathcal{A})}$; moreover, we show that this assignment is an equivalence between the category of conditional sets and conditional functions, and the category of non-empty sets and functions within $V^{(\mathcal{A})}$. We review the different elements of conditional set theory and study their interpretations within $V^{(\mathcal{A})}$: we show that conditional binary relations, conditional topologies, conditional metric spaces and so on can respectively be interpreted as binary relations, topologies, metric spaces and so on within $V^{(\mathcal{A})}$. All this shows that conditional set theory is a Boolean-valued interpretation of classical set theory. Then, as application of the Boolean-valued transfer principle, the conditional versions of classical theorems are automatically true without the necessity of a step-by-step proof.

As instances of application of this method, we provide conditional analogues of the Brouwer's fixed point theorem, the open mapping theorem, the Baire Category theorem, the Uniform Boundedness principle, Eberlein-Šmulian theorem, Amir-Lindenstrauss theorem, and versions of James' compactness theorem. In addition, by using the methods of conditional set theory we provide step-by-step proofs (not derived from the transfer principle) of some of these theorems, which can be found in the Appendix at the end of this work together to a comprehensive study of conditional weak topologies (see Appendix B).

Chapter 4: Robust representation of conditional risk measures: In this chapter, we establish a Boolean-valued transfer principle from duality theory of conventional risk measures to duality theory of conditional risk measures; namely, we develop a method to reinterpret theorems on dual representation of classical risk measures as new theorems on dual representation of conditional risk measures.

Let us put this problem in context. Suppose that the market information at some time horizon $T > 0$ is modelled by the probability space $(\Omega, \mathcal{E}, P)$, and consider a solid subspace $\mathcal{X}$ of $L^0(\mathcal{E})$ which contains the constant functions. The space $\mathcal{X}$ describes all possible final payoffs. The riskiness of any $x \in \mathcal{X}$ is quantified by a function $\rho : \mathcal{X} \to \mathbb{R}$ which satisfies:

1. *Convexity:* $\rho(rx + (1-r)y) \leq r\rho(x) + (1-r)\rho(y)$ for all $r \in \mathbb{R}$ with $0 \leq r \leq 1$;
2. *Monotonicity:* If $x \leq y$, then $\rho(y) \leq \rho(y)$;
3. *Cash invariance:* If $r \in \mathbb{R}$, then $\rho(x + r) = \rho(x) - r$.

Such a function is called *convex risk measure.* The Köthe dual space of $\mathcal{X}$ is defined to be

$$\mathcal{X}^x := \{y \in L^0(\mathcal{E}) \colon E_P[|xy|] < \infty, \, \forall x \in \mathcal{X}\}.$$

Classically, duality theory of risk measures study when a convex risk measure ρ admits a representation

$$\rho(x) = \sup\{E_P[xy] - \alpha(y) \colon y \in \mathcal{X}^x\} \quad \text{for all } x \in \mathcal{X},$$

for some penalty function $\alpha : \mathcal{X} \to \mathbb{R} \cup \{\infty\}$. Also, it is studied when the dual representation above is attained, that is,

$$\rho(x) = \max\{E_P[xy] - \alpha(y) \colon y \in \mathcal{X}^x\} \quad \text{for all } x \in \mathcal{X}.$$

For instance, for convex risk measures defined on the space L^∞ of bounded random variables, the representation with supremum is equivalent to the order lower semi-continuity of ρ, also called Fatou property (see [28, 40]). In this case, Jouini, Schachermayer and Touzi [68] proved that turning the supremum into a maximum is equivalent to the order continuity (Lebesgue property) of ρ, which is also equivalent to the weak compactness of the sub-level sets of the penalty function α.

Suppose now that $\mathcal{F} \subset \mathcal{E}$ is a sub-σ-algebra which encodes the available market information at some intermediate time $0 < t < T$. Now, the final payoffs are going to be modelled by a stable solid $L^0(\mathcal{F})$-submodule $\mathscr{X}$ of $L^0(\mathcal{E})$ which contains $L^0(\mathcal{F})$. Then a conditional risk measure is a function $\rho : \mathscr{X} \to L^0(\mathcal{F})$ satisfying for all $x, y \in \mathscr{X}$:

1. $L^0(\mathcal{F})$-*convexity:* $\rho(\eta x + (1-\eta)y) \leq r\rho(x) + (1-r)\rho(y)$ for all $\eta \in L^0(\mathcal{F})$ with $0 \leq \eta \leq 1$;
2. *Monotonicity:* $x \leq y$ implies $\rho(y) \leq \rho(x)$;
3. $L^0(\mathcal{F})$-*cash invariance:* $\rho(x+\eta) = \rho(x) - \eta$ for all $\eta \in L^0(\mathcal{F})$.

This type of risk measure was introduced by Filipovic et al. [37].

We define the Köthe dual $L^0(\mathcal{F})$-module of $\mathscr{X}$ to be

$$\mathscr{X}^x := \{y \in L^0(\mathcal{E}) \colon E_P[|xy||\mathcal{F}] < \infty, \, \forall x \in \mathscr{X}\},$$

where $E_P[\cdot|\mathcal{F}]$ denotes the extended conditional expectation.

Let $\mathcal{A} := \bar{\mathcal{F}}$ the probability algebra associated to $\mathcal{F}$ and consider the corresponding Boolean-valued model $V^{(\mathcal{A})}$. In this chapter, we prove that $L^0(\mathcal{E})$ can be interpreted as a space $L^0(\mathcal{E})\!\uparrow$ of random variables within the model $V^{(\mathcal{A})}$; any stable solid $L^0(\mathcal{F})$-submodule $\mathscr{X} \subset L^0(\mathcal{E})$ can be interpreted as a solid subspace $\mathscr{X}\!\uparrow \subset L^0(\mathcal{E})\!\uparrow$ within $V^{(\mathcal{A})}$; and, moreover, any conditional risk measure $\rho : \mathscr{X} \to L^0(\mathcal{F})$ can be viewed as a (one-period) convex risk measure $\rho\!\uparrow : \mathscr{X}\!\uparrow \to \mathbb{R}$ within $V^{(\mathcal{A})}$. Due to the transfer principle of $V^{(\mathcal{A})}$, $\rho\!\uparrow$ satisfies any theorem on dual representation of convex risk measures, thus, any of that theorems can be reinterpreted as a statement which is satisfied by ρ. As a consequence, we have that any theorem on dual representation of convex risk measures gives rise to a new theorem on dual representation of conditional risk measures. This amounts to a Boolean-valued transfer principle from duality theory of convex risk measures to duality theory of conditional risk measures.

By means of this method, we establish a general robust representation theorem which characterizes when a conditional risk measure $\rho : \mathscr{X} \to L^0(\mathcal{F})$ has a dual representation

$$\rho(x) = \sup\{E_P[xy|\mathcal{F}] - \alpha(y) \colon y \in \mathscr{X}^x\} \quad \text{for all } x \in \mathscr{X},$$

(see Theorem 4.3.1) and characterizes, in terms of a suitable compactness condition, when the representation above is attained (see Theorem 4.3.2), that is,

$$\rho(x) = \max\{E_P[xy|\mathcal{F}] - \alpha(y) \colon y \in \mathscr{X}^x\} \quad \text{for all } x \in \mathscr{X}.$$

Filipovic et al. [37] proposed the so-called L^p type module as model spaces for conditional risk measures. Namely, given $1 \leq p \leq \infty$, the corresponding L^p type module is define by

$$L^p_{\mathcal{F}}(\mathcal{E}) := \{x \in L^0(\mathcal{E}) \colon E_P[|x|^p|\mathcal{F}] < \infty\}.$$

We prove that $L^p_{\mathcal{F}}(\mathcal{E})$ can in fact be interpreted as a classical L^p space within the model $V^{(\mathcal{A})}$. Moreover, we show that the same can be done for module analogues of the classical Orlicz and Orlicz-heart spaces. For any of these cases, we establish a particular robust representation theorem for conditional risk measures (see Theorems 4.3.3, 4.3.4, 4.3.5, 4.3.6, 4.3.7 and 4.3.8).

Chapter 5: Applications to optimal stochastic control: This chapter investigates parameter-dependent stochastic optimization in finite discrete time with the tools of L^0-convex analysis. In the following, we introduce the mathematical problem and sketch the solution strategy. Suppose that we have a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t=0}^T, P)$. Consider a *forward generator* $(v_t)_{t=0}^{T-1}$ which drives a $\mathbb{R}^d$-valued adapted stochastic process

$$x_{t+1} = v_t(x_t, z_t)$$

the dynamic of which depends on a $\mathbb{R}^n$-valued adapted process (z_t) modelling the decisions chosen recursively in a state-dependent control set $\Theta_t(x_t)$ for each $t = 0, \ldots, T-1$, satisfying suitable measurable assumptions. Given a *backward generator* $(u_t)_{t=0}^T$, the goal is to maximize

$$u_0(x_0, \cdot, z_0) \circ \ldots \circ u_{T-1}(x_{T-1}, \cdot, z_{T-1}) \circ u_T(x_T) \tag{2}$$

over all possible forward processes $(x_t)_{t=0}^T$ initialized at some initial point x_0, where $\circ$ denotes composition of functions. By the Bellman principle, the global stochastic optimization problem (2) is solved by a backward recursion if all the local one-period problems

$$\begin{aligned} y_t(x_t) =& \sup_{z_t \in \Theta_t(x_t)} u_t(x_t, y_{t+1}(v_t(x_t, z_t)), z_t), \quad t = 0, \ldots, T-1 \\ y_T(x_T) =& u_T(x_T) \end{aligned}$$

attain their maxima. Each local problem at time t is the optimization of a function

$$y_t : L^0(\mathcal{F}_t, \mathbb{R}^n) \longrightarrow L^0(\mathcal{F}_t, \mathbb{R} \cup \{-\infty\}).$$

To find its extremum, we use a module analogue of the elementary fact that a semi-continuous real functional on a compact set attains its extremum. Specifically, since we work in finite dimension, it is sufficient that each local function y_t satisfies pointwise almost sure sequential semi-continuity on a sequentially closed and $L^0(\mathcal{F}_t)$-norm bounded subset of $L^0(\mathcal{F}_t, \mathbb{R}^d)$.

We provide two theorems on the existence of optimal solutions of the stochastic control problem above. Namely, in Theorem 5.2.1, we rely on compactness conditions on the control sets $\Theta_t(x_t)$. In Theorem 5.2.3, we do not assume constraints on the controls, but assume stronger assumptions on the forward and backward generators.

Classically, the tool used to study optimal stochastic control are *normal integrands*, see eg [13, 91, 92, 94, 95, 96, 97] and for an introduction, we refer to [98]. In this chapter, we prove that there is a one-to-one correspondence between normal integrands from $\Omega \times \mathbb{R}^d$ to $\mathbb{R} \cup \{\infty\}$ and sequentially semi-continuous functions from $L^0(\mathcal{F}_t, \mathbb{R}^d)$ to $L^0(\mathcal{F}_t, \mathbb{R} \cup \{\infty\})$ with the local property.

We study some related financial problems: We find optimal solutions for a problem of wealth-dependent utility maximization problem under short-selling constraints and self-financing strategies; we study wealth-dependent utility maximization problems for generalized entropic utility functions; and we find an explicit solution for a utility sharing problem with entropic utility functions.

Finally, we end this chapter with a new proof based on L^0-convex analysis for the Dalang-Morton-Willinger theorem, which is the finite-time version of the celebrated First Fundamental Theorem of Asset Pricing (see [30]).

Related publications

The content of this thesis corresponds to that of the following papers:

Papers accepted in peer-reviewed journals:

[8] A. Avilés, J.M.Z. *Boolean-valued models as a foundation for locally L^0-convex analysis and Conditional set theory*, Journal of Applied Logics, 5(1): 389-420, 2018.

[88] J. Orihuela, J.M.Z. *Stability in locally L^0-convex modules and a conditional version of James' compactness theorem.* Journal of Mathematical Analysis and Applications, 452(2): 1101-1127, 2017.

[115] J.M.Z. *Versions of Eberlein-Šmulian and Amir-Lindenstrauss theorems in the framework of conditional sets.* Applicable Analysis and Discrete Mathematics, 10(2): 231-261, 2016.

[117] J.M.Z. *On the Characterization of Locally L^0-Convex Topologies Induced by a Family of L^0-Seminorms.* Journal of Convex Analysis, 24(2): 383-391, 2017.

[118] J.M.Z. *Randomized versions of Mazur lemma and Krein-Šmulian Theorem.* Journal of Convex Analysis, 25(2), 2018. (To appear).

Preprints:

[63] A. Jamneshan, M. Kupper, J.M.Z. *Parameter-dependent Stochastic Optimal Control in Finite Discrete Time*, Arxiv preprint, 2017. (Passed a round of review in SIAM Journal on Control and Optimization).

[64] A. Jamneshan, J.M.Z. *On compactness in topological L^0-modules*, Arxiv preprint, 2017.

[116] J.M.Z., *A Boolean-valued model approach to conditional risk*, Arxiv preprint, 2017.

For the sake of clarity in the exposition and homogeneity in the set-up of the thesis, the organization and presentation of the results in this thesis does not exactly correspond that of the listed papers:

The content of Chapter 1 corresponds essentially to that of [117], [118] and parts of [88] and [64]. The content of Chapters 2 and 3 covers that of [8]. The content of Chapter 4 corresponds essentially to that of [116]. The content of Chapter 5 covers that of [63].

Historically, some results on conditional set theory were proven by conventional means during the first stages of the researching period preceding this work. Since these results turned out to be consequences of the Boolean-valued transfer principle established in Chapter 3, their step-by-step proofs have been placed in the Appendix at the end of this work (see Appendix B). This appendix covers the content of our publication [115] and part of the content of [88].

Setup and notational conventions

This work lies within the framework of the Zermelo-Fraenkel set theory together with the axiom of choice, which is denoted by the acronym ZFC (for a review of ZFC set theory, see the Appendix A). Throughout this thesis we will consider an underlying probability space $(\Omega, \mathcal{F}, P)$. We denote by $(\bar{\mathcal{F}}, \bar{P})$ the probability algebra associated to $(\Omega, \mathcal{F}, P)$, where $\bar{\mathcal{F}}$ is defined by identifying events whose symmetric difference has probability 0 and $\bar{P}(a) := P(A)$ where $A \in \mathcal{F}$ is any representative of $a \in \bar{\mathcal{F}}$. For the sake of brevity, many times we call $\bar{\mathcal{F}}$ the probability algebra associated to $\mathcal{F}$. We denote the equivalence classes in $\bar{\mathcal{F}}$ by the lower case latin letters a, b, c and their indexed forms. If we use A and a, B and b, etc., then it is always understood that A is a representative of a, B is a representative of b, etc. The class of $\emptyset$ is denoted by 0 and the class of Ω is denoted by I. It is satisfied that $\bar{\mathcal{F}}$ has structure of complete Boolean algebra which satisfies the *countable chain condition* (ccc), that is, every pairwise family in $\mathcal{A}$ is at most countable (a more detailed explanation on Boolean algebras and probability algebras can be found in Section A.6). In general, if $\mathcal{A}$ is a complete Boolean algebra, we denote the supremum and infimum of a family $(a_i)_{i\in\Delta}$ of elements of $\mathcal{A}$ by $\bigvee_{i\in\Delta} a_i$ and by $\bigwedge_{i\in\Delta} a_i$, respectively. If $a \in \mathcal{A}$, we denote by $\mathcal{A}_a := \{b \in \mathcal{A} \colon b \leq \mathcal{A}\}$ the ideal of $\mathcal{A}$ generated by a. Suppose that $a \in \mathcal{A}$, a family $(a_i)_{i\in\Delta}$ in $\mathcal{A}$ is said to be a partition of a to $\mathcal{A}$ if $\bigvee_{i\in\Delta} a_i = a$ and $a_i \wedge a_j = 0$ for all $i, j \in \Delta$ with $i \neq j$. Given $a \in \mathcal{A}$, $p(a) := p_{\mathcal{A}}(a)$ stands for the set of all partitions of a to $\mathcal{A}$. In the case that $\mathcal{A} = \bar{\mathcal{F}}$, we have that any partition is at most countable due to the ccc.

We denote by $L^0 := L^0(\mathcal{F})$ the set of real valued $\mathcal{F}$-measurable random variables modulo almost sure equality. It is known that the triple $\left(L^0, +, \cdot\right)$ endowed with the partial order of almost sure dominance is a Dedekind complete lattice ordered ring. Given $a \in \bar{\mathcal{F}}$, 1_a stands for the class in L^0 of the characteristic function of some representative A of a. For given $\eta, \xi \in L^0$, we define

$$\{\eta = \xi\} := \bigvee\{a \in \bar{\mathcal{F}} \colon 1_a\eta = 1_a\xi\};$$

$$\{\eta \leq \xi\} := \bigvee\{a \in \bar{\mathcal{F}} \colon 1_a\eta \leq 1_a\xi\};$$

$$\{\eta < \xi\} := \{\eta \leq \xi\} \wedge \{\eta = \xi\}^c;$$

also, if $a \in \bar{\mathcal{F}}$, we write $\eta \leq \xi$ (resp. $\eta < \xi$) on a if $a \leq \{\eta \leq \xi\}$ (resp. $a \leq \{\eta < \xi\}$). If $\eta \in L^0$, we use the notation $\eta^+ := \eta \vee 0$, $\eta^- := (-\eta)^+$ and $\operatorname{sgn}(\eta) := 1_{\{\eta\geq 0\}} - 1_{\{\eta<0\}}$. Suppose that (η_n) is a sequence in L^0 and $\eta \in L^0$, we write $\lim_n \eta_n = \eta$ a.s. in L^0 whenever η_n converges almost surely to η (equivalently, η_n order converges to η). If (η_n) is a decreasing (resp. increasing) sequence, i.e. $\eta_n \geq \eta_{n+1}$ (resp. $\eta_n \leq \eta_{n+1}$) and $\lim_n \eta_n = \eta$ a.s. in L^0, we write $\eta_n \searrow \eta$ a.s. (rep. $\eta_n \nearrow \eta$ a.s.). We define $\limsup_n \eta_n := \inf_n \sup_{m\geq n} \eta_m$ and $\liminf_n \eta_n := \sup_n \inf_{m\geq n} \eta_m$.

We use the notation

$$L^0_+ := L^0_+(\mathcal{F}) = \left\{\eta \in L^0 \colon \eta \geq 0\right\},$$

$$L^0_{++} := L^0_{++}(\mathcal{F}) = \left\{\eta \in L^0 \colon \eta > 0\right\}.$$

Also, we denote by $L^0(\mathbb{N}) := L^0(\mathcal{F}, \mathbb{N})$ the set of equivalence classes of $\mathcal{F}$-measurable random variables taking values in $\mathbb{N}$. The members of $L^0(\mathbb{N})$ are denoted by $\mathfrak{n}, \mathfrak{m}, \ldots$. In general, if H is a subset of $\mathbb{R}^d$ for some natural number d, we denote by $L^0(H) := L^0(\mathcal{F}, H)$ the space of $\mathcal{F}$-measurable random variables taking values in H.

By $\bar{L}^0 := \bar{L}^0(\mathcal{F})$ we denote the set of equivalence classes of $\mathcal{F}$-measurable random variables taking values in $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. In addition, we follow the convention $0 \cdot \infty = 0$. If $H \subset \overline{\mathbb{R}}$, we denote by $\bar{L}^0(H) := \bar{L}^0(\mathcal{F}, H)$ the space of $\mathcal{F}$-measurable random variables taking values in H.

In some cases, we will consider a probability space $(\Omega, \mathcal{E}, P)$ where $\mathcal{F}$ is a sub-σ-algebra of $\mathcal{E}$. We will denote by $x, y, z, \ldots$ the elements of $L^0(\mathcal{E})$. The conditional expectation $E_P[\cdot|\mathcal{F}]$ is classically defined on $L^1(\mathcal{E})$. Suppose that $x \in L^0(\mathcal{E})$ satisfies that

$$\{\lim_n E_P[x^+ \wedge n|\mathcal{F}] = \infty\} \wedge \{\lim_n E_P[x^- \wedge n|\mathcal{F}] = \infty\} = 0,$$

then we define the *extended conditional expectation* of x to be

$$E_P[x|\mathcal{F}] := \lim_n E_P[x^+ \wedge n|\mathcal{F}] - \lim_n E_P[x^- \wedge n|\mathcal{F}] \in \bar{L}^0(\mathcal{F}).$$

In some cases, we will consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, P)$. If $H \subset \mathbb{R}^d$, we use the notation $L^0_t(H) := L^0(\mathcal{F}_t, H)$, and if $H \subset \overline{\mathbb{R}}$, we define $\bar{L}^0_t(H) := \bar{L}^0(\mathcal{F}_t, H)$ for $t = 0, \ldots, T$. For any $t = 0, \ldots, T$, we can consider the probability algebra associated to $\mathcal{F}_t$, which is denoted by $\bar{\mathcal{F}}_t$. For given $a \in \bar{\mathcal{F}}_t$, the record $p_t(a)$ stands for the set of partitions of a to $\bar{\mathcal{F}}_t$.

In general, if (X, τ) is a topological space and C is a subset of X we denote by $\overline{C}$ the topological closure of C and by $\overset{o}{C}$ the topological interior.

Chapter 1

A conventional approach to L^0-convex analysis

In this chapter, we study some open questions on L^0-modules by means of several results and examples of boundary L^0-modules.

In Section 1.1, we start by studying some algebraic aspects on L^0-modules. We review the notion of countable concatenation and discuss existence and uniqueness. The notion of stabilization of an L^0-module is introduced. We prove that a free L^0-module has the countable concatenation property if and only if it is finitely generated. We study L^0-modules with additive class of stable subsets.

In Section 1.2, a characterization of locally L^0-convex topologies induced by a family of L^0-seminorms is established. In addition, we provide a counter-example of a locally L^0-convex topology which is not induced for any family of L^0-seminorms.

In Section 1.3, we prove a module analogue of the classical Mazur's theorem for L^0-normed modules with additive class of stable subsets.

In Section 1.4 we introduce the notion of locally stable L^0-convex topology, that is, a locally L^0-convex topology that admits a stable neighborhood base of the origin. We provide examples of locally L^0-convex topologies which are not locally stable L^0-convex, for instance, we show that the module weak and weak-$*$ topologies are not stable in general, and we introduce the locally stably weak and weak-$*$ topologies. We characterize locally stable L^0-convex topologies in terms of inducing families of L^0-seminorms. We prove that a locally stable L^0-convex module is anti-compact (i.e. any compact subset is finite) if and only if its support is atomless.

Finally, in Section 1.5, we study relations among three types of L^0-module topologies: locally L^0-convex topologies, locally L^0-convex topologies and the so-called (ϵ, λ)-topologies.

1.1 Countable concatenation property and stability

The *countable concatenation property* is a technical assumption on L^0-modules which is used in the literature of L^0-convex analysis, see [36, 51]. Although the first results of this thesis are proven without the necessity of this property, the countable concatenation property will be crucial to establish the systematic approach that will be developed in Chapter 2.

Suppose that E is an L^0-module:

- given a partition $(a_k) \in p(I)$ and a sequence (x_k) in E, a member x of E is called a *countable concatenation of* (x_k) *along* (a_k) whenever

$$1_{a_k}x = 1_{a_k}x_k \quad \text{for all } k \in \mathbb{N}.$$

- E is said to have the *countable concatenation property* if for every partition $(a_k) \in p(I)$ and sequence (x_k) in E there exists a unique concatenation $x \in E$ of (x_k) along (a_k).

If E has the countable concatenation property, the unique concatenation of a sequence (x_k) in E along the partition $(a_k) \in p(I)$ is denoted by $\sum_{k\in\mathbb{N}} 1_{a_k}x_k$, or simply $\sum 1_{a_k}x_k$.

One could think that every L^0-module has the countable concatenation property. However, we will see next two examples of L^0-modules without this property. In the first one, the existence of countable concatenations fails and, in the second one, the uniqueness is not satisfied.

Example 1.1.1. *[36, Example 2.12] Take* $\Omega = (0,1)$*,* $\mathcal{F} = \mathcal{B}(\Omega)$ *the Borel* σ*-algebra,* $A_k = [\frac{1}{2^k}, \frac{1}{2^{k-1}})$ *with* $k \in \mathbb{N}$ *and* $P := \lambda$ *the Lebesgue measure. Then, the* L^0*-module*

$$E := \operatorname{span}_{L^0}\{1_{a_k} : k \in \mathbb{N}\}$$

does not have the countable concatenation property. For instance, it is impossible to find a concatenation in E *of the sequence* (1_{a_k}) *along the partition* (a_k)*.*

Regarding the failure of the uniqueness of concatenations we provide the following example:

Example 1.1.2. *Take* $\Omega = (0,1)$*,* $\mathcal{F} = \mathcal{B}(\Omega)$ *the Borel* σ*-algebra,* $A_k = [\frac{1}{2^k}, \frac{1}{2^{k-1}})$ *with* $k \in \mathbb{N}$ *and* $P := \lambda$ *the Lebesgue measure. We define in* $L^0 := L^0(\Omega, \mathcal{F}, P)$ *the following equivalence relation*

$$x \sim y \quad \text{if } 1_{a_k}x = 1_{a_k}y \quad \textit{for all but finitely many } k \in \mathbb{N}.$$

If we denote by $\overline{x}$ *the equivalence class of* x*, we can define* $\overline{x} + \overline{y} := \overline{x+y}$ *and* $\eta \cdot \overline{x} := \overline{\eta x}$*, obtaining that the quotient set* $E := L^0/\sim$ *is an* L^0*-module.*

Then, for any $\bar{x} \in E$*, we have that* $1_{a_k}\overline{x} = \overline{1_{a_k}x} = \overline{0}$ *for each* $k \in \mathbb{N}$*. Hence, any element of* E *is a concatenation of* $(0)_k$ *along* $(a_k)_k$*.*

As shown, not every L^0-module has the countable concatenation property. However, given any L^0-module E, we can construct from it another L^0-module $E_{\mathfrak{s}}$ with the countable concatenation property. Indeed, we consider the set of all sequences $(x_k, a_k)_{k\in\mathbb{N}}$ in $E \times \mathcal{A}$, where $(a_k) \in p(I)$, and the equivalence relation

$$(x_i, a_i)_i \sim (y_j, b_j)_j \quad \text{whenever} \quad 1_{a_i\wedge b_j}x_i = 1_{a_i\wedge b_j}y_j \text{ for all } i, j \in \mathbb{N}.$$

Given $(x_k, a_k)_k$ with $(a_k) \in p(I)$ and $(x_k) \subset E$, we will denote by $[x_k, a_k]_k$ its class of equivalence, which is referred to as the *stratification of the sequence* (x_k) *along the partition* (a_k).

Let $E_{\mathfrak{s}}$ denote the sets of all stratifications of sequences $(x_k) \subset E$ along partitions $(a_k) \in p(I)$. For $[x_i, a_i]_i, [y_j, b_j]_j \in E_{\mathfrak{s}}$ and $\eta \in L^0$, the operations

$$[x_i, a_i]_i + [y_j, b_j]_j := [x_i + y_j, a_i \wedge b_j]_{i,j} \quad \text{and} \quad \eta[x_i, a_i]_i := [\eta x_i, a_i]_i$$

are well-defined and $E_{\mathfrak{s}}$ endowed with them is an L^0-module. Moreover, it is simple to verify that $E_{\mathfrak{s}}$ has the countable concatenation property, where for any sequence $([x_i^k, a_i^k]_i)_k$ in $E_{\mathfrak{s}}$ and partition $(b_k) \in p(I)$ one has

$$\sum_{k\in\mathbb{N}} 1_{b_k}[x_i^k, a_i^k]_i = [x_i^k, b_k \wedge a_i^k]_{i,k}.$$

The L^0-module $E_{\mathfrak{s}}$ will be referred to as the *stabilization* of E.

Throughout, we will suppose that E is an L^0-module which does not have necessarily the countable concatenation property.

Recall that E is a *free module* if it has a non-empty algebraic basis and that every finitely generated free module has a finite basis (see eg [2]).

The following result characterizes when a free L^0-module has the countable concatenation property:

Proposition 1.1.1. *Suppose that Ω is not the union of finitely many atoms. Let E be a free L^0-module. Then E has the countable concatenation property if and only if E is finitely generated.*

Proof. Suppose that E is finitely generated. Then E has a basis $\{x_1, \ldots, x_n\}$. Let $(a_k) \in p(I)$ and $(y_k) \subset E$. Each y_k is of the form $\eta_1^k x_1 + \ldots + \eta_n^k x_n$ with $\eta_m^k \in L^0$ for $1 \leq m \leq n$. Put $\eta_m := \sum 1_{a_k} \eta_m^k$ for $m = 1, \ldots, n$ and $y := \eta_1 x_1 + \ldots + \eta_m x_m$. Then $1_{a_k} y = 1_{a_k} y_k$ for all k. Now, let $z \in E$ be satisfying $1_{a_k} z = 1_{a_k} y_k$ for all k. If we prove that $y = z$, then we conclude that E has the countable concatenation property. Indeed, z is of the form $\xi_1 x_1 + \ldots + \xi_n x_n$ with $\xi_m \in L^0$ for $1 \leq m \leq n$. Then, for each k, $0 = 1_{a_k}(y - z) = 1_{a_k}(\eta_1 - \xi_1)x_1 + \ldots + 1_{a_k}(\eta_n - \xi_n)x_n$. It follows that $1_{a_k}(\eta_1 - \xi_1) = \ldots = 1_{a_k}(\eta_n - \xi_n) = 0$ for each k. Thus $\eta_m = \xi_m$ for $m = 1, \ldots, n$. We conclude that $y = z$.

Conversely, suppose that E has the countable concatenation property. By way of contradiction, suppose that E is not finitely generated. Since E is free, E has a basis of not determined length $(x_i)_{i \in I}$. Since Ω is not the union of finitely many atoms, we can find an infinite partition $(a_k) \in p(I)$. Let (i_k) be an injective sequence in I and let $x := \sum 1_{a_k} x_{i_k}$. Then there exists a finite set $J \subset I$ and $(\eta_i)_{i \in J}$ such that $x = \sum_{i \in J} \eta_i x_i$. Since J is finite, we can choose $i_k \in I \setminus J$ for some $k \in \mathbb{N}$ with $a_k > 0$. One has

$$0 = 1_{a_k} x - 1_{a_k} x = \sum_{i \in J} \eta_i 1_{a_k} x_i - 1_{a_k} x_{i_k}.$$

Then necessarily $1_{a_k} = 0$, but this contradicts $a_k > 0$. $\square$

Let us introduce more concepts:

- A non-empty subset S of E is said to be *stable*, if $x \in S$ whenever $x \in E$ and there is a sequence $(x_k) \subset S$ and a partition $(a_k) \in p(I)$ such that $1_{a_k} x = 1_{a_k} x_k$ for all $k \in \mathbb{N}$ (i.e. $[x, a_k]_k = [x_k, a_k]_k$).
- Given a non-empty subset S of E we define its *stable hull* as follows

$$\mathfrak{s}(S) := \{x \in E \colon \exists (x_k) \subset S,\ (a_k) \in p(I) \text{ such that } 1_{a_k} x = 1_{a_k} x_k,\ \forall k \in \mathbb{N}\}.$$

Note that these definitions make sense when E does not have the countable concatenation property. If S is a non-empty subset of E, then $\mathfrak{s}(S)$ is the smallest stable subset containing S. Note that any L^0-module is a stable subset of itself (even if E does not have the countable concatenation property).

Example 1.1.3. *Take $\Omega = (0, 1)$, $\mathcal{F} = \mathcal{B}(\Omega)$ the Borel σ-algebra, $A_k = [\frac{1}{2^k}, \frac{1}{2^{k-1}})$ with $k \in \mathbb{N}$ and $P := \lambda$ the Lebesgue measure. Let $E := \operatorname{span}_{L^0}\{1_{a_k} \colon k \in \mathbb{N}\}$.*

The set $S := \{x \in E \colon |x| < 1\}$ is a stable subset of E. However, E does not have the countable concatenation property.

Remark 1.1.1. *Note that, if E has the countable concatenation property, then S is stable if and only if for any sequence $(x_k) \subset S$ and partition $(a_k) \in p(I)$ it is satisfied that $\sum_k 1_{a_k} x_k \in S$.*

Moreover, if S is a non-empty subset of E, we have that

$$\mathfrak{s}(S) := \left\{ \sum_k 1_{a_k} x_k \colon (a_k) \in p(I),\ (x_k) \subset S \right\}.$$

Another property that we are interested in is the following:

Definition 1.1.1. *We say that an L^0-module has* additive class of stable subsets *if for any two stable subsets S_1, S_2 of E we have that $S_1 + S_2 := \{x + y \colon x \in S_1,\ y \in S_2\}$ is again a stable subset.*

The following example shows that, in general, the sum of two stable subsets is not stable.

Example 1.1.4. *Let $\Omega = (0,1)$, $A_k = [\frac{1}{2^k}, \frac{1}{2^{k-1}})$ for each $k \in \mathbb{N}$, $\mathcal{F} = \sigma(\{A_k \colon k \in \mathbb{N}\})$ the σ-algebra generated by $\{A_k \colon k \in \mathbb{N}\}$, and $P = \lambda$ the Lebesgue measure restricted to $\mathcal{F}$. Consider the following L^0-submodule of $L^0(\mathcal{F}, \mathbb{R}^2)$:*

$$E := \mathrm{span}_{L^0}\{(1,0), (0, 1_{a_k}) \colon k \in \mathbb{N}\},$$

and set

$$L := \mathfrak{s}\left(\{(0, -1_{a_k}) \colon k \in \mathbb{N}\}\right), \quad M := \mathfrak{s}\left(\{(1_{a_k}, 1_{a_k}) \colon k \in \mathbb{N}\}\right),$$

which are obviously stable.

We have that $(1,0) \in E$ and

$$1_{a_k}(1,0) = 1_{a_k}(1_{a_k}(0, -1_{a_k}) + (1_{a_k}, 1_{a_k})) \in 1_{a_k}(L + M) \quad \text{for all } k \in \mathbb{N},$$

hence $(1,0) \in \mathfrak{s}(L + M)$.

However, a close look shows that $(1,0) \notin L + M$. We conclude that $L + M$ is not stable.

A simple verification shows that every L^0-module with the countable concatenation property has additive class of stable subsets. However, the next result exhibits that the class of L^0-modules with the former property is strictly smaller than the class of L^0-modules with the latter property.

Proposition 1.1.2. *There exists an L^0-module with additive class of stable subsets but without the countable concatenation property.*

Proof. Let $\Omega = (0,1)$, $A_k = [\frac{1}{2^k}, \frac{1}{2^{k-1}})$ for each $k \in \mathbb{N}$, $\mathcal{F} = \sigma(\{A_k \colon k \in \mathbb{N}\})$, and $P = \lambda$ Lebesgue measure restricted to $\mathcal{F}$. Take $E := \mathrm{span}_{L^0}\{1_{a_k} \colon k \in \mathbb{N}\}$. We know that E does not have the countable concatenation property (see Example 1.1.1). Let us prove that E has additive class of stable subsets. Indeed, suppose that L, M are stable subsets of E. Let $z \in E$, $(b_k) \in p(I)$, $(x_k) \subset L$ and $(y_k) \subset M$ be with

$$1_{b_k} z = 1_{b_k}(x_k + y_k) \quad \text{for all } k \in \mathbb{N},$$

and show that $z \in L + M$.

First, note that the set

$$F_L := \{i \in \mathbb{N} \colon 1_{a_i} x \neq 0,\ \forall x \in L\}$$

is necessarily finite. Define F_M in an analogous way, which is finite either.

There exists a finite subset F of $\mathbb{N}$ such that $z = \sum_{i\in F} 1_{a_i} z_i$ with $z_i \neq 0$. For any $i \in F$, since a_i is an atom, there exists k_i so that $a_i \leq b_{k_i}$. Necessarily we have

$$\begin{aligned} z =& \sum_{i\in F} 1_{a_i}(x_{k_i} + y_{k_i}) \\ =& \sum_{i\in F} 1_{a_i}(x_{k_i} + y_{k_i}) + \sum_{i\in (F_L\cup F_M)\setminus F} 1_{a_i}(x_{k_i} + y_{k_i}) \\ =& x + y, \end{aligned}$$

where

$$x := \sum_{i\in F\cup F_L\cup F_M} 1_{a_i} x_{k_i} \quad \text{and} \quad y := \sum_{i\in F\cup F_L\cup F_M} 1_{a_i} y_{k_i}.$$

Since $F \cup F_L \cup F_M$ is finite, we have that $x, y \in E$.

Further, fixed $i \in \mathbb{N}$, if $i \in F \cup F_L \cup F_M$, then $1_{a_i} x = 1_{a_i} x_i$ and $x_i \in L$. If $i \notin F \cup F_L \cup F_M$, in particular $i \notin F_L$, hence there exists $x_i \in L$ with $1_{a_i} x_i = 0 = 1_{a_i} x$. Summing up, we have that $1_{a_i} x \in 1_{a_i} L$ for all $i \in \mathbb{N}$. Since L is stable, it follows that $x \in L$. Similarly, $y \in M$.

We conclude that $L + M$ is stable. □

1.2 Characterization of locally L^0-convex topologies induced by families of L^0-seminorms

The order of almost sure dominance allows us to define a topology on L^0. Namely, let $B_\varepsilon := \left\{\eta \in L^0 \colon |\eta| \leq \varepsilon\right\}$ be the ball of radius $\varepsilon \in L^0_{++}$ centered at $0 \in L^0$. Then, $\mathscr{U} := \left\{\eta + B_\varepsilon \colon \eta \in L^0,\, \varepsilon \in L^0_{++}\right\}$ is a neighborhood base of a Hausdorff topology on L^0 (see Filipovic et al. [36]).

Let us recall some notions of the theory of locally L^0-convex modules:

- [36, Definition 2.1] A *topological L^0-module* $(E, \mathscr{T})$ is an L^0-module E endowed with a topology $\mathscr{T}$ such that

$$E \times E \longrightarrow E,\ (x, y) \mapsto x + y, \quad L^0 \times E \longrightarrow E,\ (\eta, x) \mapsto \eta x$$

  are continuous with the corresponding product topologies.

- [36, Definition 2.2] A *locally L^0-convex module* is a topological L^0-module $(E, \mathscr{T})$ which admits a neighborhood base $\mathscr{U}$ of $0 \in E$ such that each $U \in \mathscr{U}$ is

  1. L^0-*convex*, i.e. $\eta x + (1-\eta) y \in U$ for all $x, y \in U$ and $\eta \in L^0$ with $0 \leq \eta \leq 1$;
  2. L^0-*absorbing*, i.e. for all $x \in E$ there is a $\eta \in L^0_{++}$ such that $x \in \eta U$;
  3. L^0-*balanced*, i.e. $\eta x \in U$ for all $x \in U$ and $\eta \in L^0$ with $|\eta| \leq 1$.

  In this case, $\mathscr{T}$ is called a *locally L^0-convex topology.*

- [36, definition 2.3] A function $\|\cdot\| : E \to L^0_+$ is an L^0-*seminorm* on E if:

  1. $\|\eta x\| = |\eta|\, \|x\|$ for all $\eta \in L^0$ and $x \in E$;
  2. $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in E$.

  If moreover, $\|x\| = 0$ implies $x = 0$, then $\|\cdot\|$ is an L^0-*norm* on E.

- A family of L^0-seminorms $\mathscr{P}$ on E is called *separated* if it satisfies:

$$x = 0 \quad \text{if and only if} \quad \sup\{\|x\|\colon \|\cdot\| \in \mathscr{P}\} = 0.$$

Let $\mathscr{P}$ be a family of L^0-seminorms on E. Given a finite subset F of $\mathscr{P}$ and $\varepsilon \in L^0_{++}$, we define

$$U_{F,\varepsilon} := \left\{ x \in E\colon \sup_{\|\cdot\| \in F} \|x\| \leq \varepsilon \right\}.$$

Then $\mathscr{U} := \left\{ U_{F,\varepsilon}\colon \varepsilon \in L^0_{++},\ F \subset \mathscr{P} \text{ finite} \right\}$ is a neighborhood base of $0 \in E$ for some locally L^0-convex topology, denoted by $\langle \mathscr{P} \rangle$, which is called the *topology induced* by $\mathscr{P}$ (see [36]).

Examples 1.2.1. *1. Filipovic et al. [36] introduced the following locally L^0-convex modules, which are called L^p type modules. Namely, let $(\Omega, \mathcal{E}, P)$ be a probability space such that $\mathcal{F}$ is a sub-σ-algebra of $\mathcal{E}$ and $p \in [1, \infty]$. Define $\|\cdot|\mathcal{F}\|_p : L^0(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ by*

$$\|x|\mathcal{F}\|_p := \begin{cases} E_P\left[|x|^p \,|\mathcal{F}\right]^{1/p}, & \text{if } p < \infty \\ \inf\left\{ \eta \in \bar{L}^0(\mathcal{F}) : \eta \geq |x| \right\}, & \text{if } p = \infty. \end{cases}$$

Then $L^p_{\mathcal{F}}(\mathcal{E}) := \{x \in L^0(\mathcal{E})\colon \|x|\mathcal{F}\|_p \in L^0(\mathcal{F})\}$ is an $L^0(\mathcal{F})$-module with the countable concatenation property and $\|x|\mathcal{F}\|_p$ is an $L^0(\mathcal{F})$-norm. Moreover, the following product structure $L^p_{\mathcal{F}}(\mathcal{E}) = L^0(\mathcal{F})L^p(\mathcal{E})$ was proven in [36] and the relation $L^p_{\mathcal{F}}(\mathcal{E}) = \mathfrak{s}(L^p(\mathcal{E}))$ was obtained in [55, Proposition 3.3].

2. Given a topological L^0-module $(E, \mathscr{T})$, in [36] it was introduced the dual L^0-module of E, which is defined to be

$$E^* := E^*[\mathscr{T}] := \left\{ \mu : E \to L^0\colon \mu \text{ is a continuous } L^0\text{-module homomorphism} \right\}.$$

Fixed $\mu \in E^*$, we define $\|\cdot\|_\mu : E \to L^0$, $\|x\|_\mu := |\mu(x)|$. It can be verified that $\|\cdot\|_\mu$ is an L^0-seminorm. We can consider the family $\{\|\cdot\|_\mu\colon \mu \in E^*\}$ of L^0-seminorms which induces a locally L^0-topology referred to as the weak topology *and denoted by $\sigma(E, E^*)$. Analogously, one can define the weak-$*$ topology $\sigma(E^*, E)$.*

Let us recall the modular version of the gauge function, which was introduced in [36]. Namely, given a subset U of E we define the gauge function of U by

$$\|\cdot\|_U : E \longrightarrow \bar{L}^0_+, \quad \|x\|_U := \inf\left\{ \eta \in L^0_{++}\colon x \in \eta U \right\}.$$

Note that the definition of $\|\cdot\|_U$ makes sense without the necessity of assuming the countable concatenation property on E. In addition, it is known that, if U is L^0-convex, L^0-absorbing, and L^0-balanced, then $\|\cdot\|_U$ is an L^0-seminorm and this holds for modules without the countable concatenation property (see [36, Proposition 2.23]).

Similarly as occurs in classical convex analysis, if one could prove that

$$\{x \in E\colon \|x\|_U < 1\} \subset U, \tag{1.1}$$

whenever $U \subset E$ is L^0-convex, L^0-absorbing and L^0-balanced, we would easily obtain that any locally L^0-convex topology is induced by a family of L^0-seminorms, just by extrapolating the classical case.

In what follows, we will show that this is not the case, and we will study which conditions are needed on U in order to obtain the relation (1.1) to be valid.

The order of almost sure dominance is not total. For this reason, the following result will be useful in some situations:

Lemma 1.2.1. *Let $C \subset L^0$ be stable and bounded below (resp. above) for the order of almost sure dominance. Then for each $\varepsilon \in L^0_{++}$ there exists $\eta_\varepsilon \in C$ such that*

$$\inf C \leq \eta_\varepsilon < \inf C + \varepsilon \quad \textit{(resp., } \sup C - \varepsilon < \eta_\varepsilon \leq \sup C\textit{)}.$$

In particular, if E is an L^0-module and $K \subset E$ is stable, L^0-convex and L^0-absorbing. Then, for every $x \in E$ and $\varepsilon \in L^0_{++}$, there exists $\eta_\varepsilon \in L^0_{++}$ with $x \in \eta_\varepsilon K$ such that

$$\|x\|_K \leq \eta_\varepsilon < \|x\|_K + \varepsilon.$$

Proof. First, let us see that C is directed downwards. Indeed, given $\eta, \xi \in C$, let $a := \{\eta < \xi\}$. Then, since C is stable, $1_a\eta + 1_{a^c}\xi = \eta \wedge \xi \in C$. Therefore, for given $\varepsilon \in L^0_{++}$, there exists a decreasing sequence $(\eta_k)_{k\in\mathbb{N}}$ in C converging to $\inf C$ almost surely.

Now, we consider the partition

$$a_0 := 0 \text{ and } a_k := \{\eta_k < \inf C + \varepsilon\} \wedge a_{k-1}^c \text{ for } k \in \mathbb{N}.$$

Let $\eta_\varepsilon := \sum_{k\in\mathbb{N}} 1_{a_k}\eta_k$. We have that $1_{a_k}\eta_\varepsilon = 1_{a_k}\eta_k$ and $\eta_k \in C$ for all $k \in \mathbb{N}$. Since C is stable, we obtain $\eta_\varepsilon \in C$.

For the second part, it suffices to see that if K is stable, then the set

$$\left\{\eta \in L^0_{++} \colon x \in \eta K\right\}$$

is a stable subset of L^0. Indeed, suppose that $(a_k) \in p(I)$ and (η_k) is a sequence in L^0_{++} such that $x \in \eta_k K$ for each $k \in \mathbb{N}$. Take $\eta := \sum_{k\in\mathbb{N}} 1_{a_k}\eta_k \in L^0_{++}$. Then we have that $1_{a_k}\eta^{-1}x = 1_{a_k}\eta_k^{-1}x$ for all $k \in \mathbb{N}$. Being K stable, we conclude that $\eta^{-1}x \in K$.

□

As a consequence of the lemma above, we obtain the following result:

Proposition 1.2.1. *Let $(E, \mathscr{T})$ be a locally L^0-convex module and let $U \subset E$ be L^0-convex, L^0-absorbing, L^0-balanced and stable. Then*

$$\{x \in E \colon \|x\|_U < 1\} \subset U \subset \{x \in E \colon \|x\|_U \leq 1\}.$$

Proof. It suffices to show that $\{x \in E \colon \|x\|_U < 1\} \subset U$. Indeed, let $x \in E$ be such that $\|x\|_U < 1$. By Lemma 1.2.1 there exists $\eta \in L^0_{++}$ such that $x \in \eta U$ and $\|x\|_U \leq \eta < 1$. Then, since U is L^0-convex, $x = \eta\frac{x}{\eta} + (1-\eta)0 \in U$, which completes the proof. □

Finally, in the theorem below we provide a characterization of the locally L^0-convex topologies that are induced by a family of L^0-seminorms.

Theorem 1.2.1. *Let $(E, \mathscr{T})$ be a topological L^0-module. Then the following conditions are equivalent:*

1. *there exists a family of L^0-seminorms $\mathscr{P}$ such that $\mathscr{T} = \langle\mathscr{P}\rangle$;*
2. *there is a neighborhood base $\mathscr{U}$ of $0 \in E$ such that each $U \in \mathscr{U}$ is stable, L^0-convex, L^0-absorbing and L^0-balanced;*
3. *there is a neighborhood base $\mathscr{U}$ of $0 \in E$ such that each $U \in \mathscr{U}$ is stable and L^0-convex.*

Proof. 1 $\Rightarrow$ 2: Suppose that $\mathscr{T} = \langle \mathscr{P} \rangle$ for some family of L^0-seminorms $\mathscr{P}$. If $F \subset \mathscr{P}$ is finite and $\varepsilon \in L^0_{++}$, it can be easily checked that $U_{F,\varepsilon}$ is L^0-convex, L^0-absorbing and L^0-balanced. Besides, $U_{F,\varepsilon}$ is stable. Indeed, suppose that $x \in E$ and there are a partition $(a_k) \in p(I)$ and a sequence $(x_k) \subset U_{F,\varepsilon}$ such that $1_{A_k} x = 1_{A_k} x_k$ for all $k \in \mathbb{N}$. Then, for $\|\cdot\| \in F$, we have

$$\|x\| = \left(\sum 1_{A_k}\right) \|x\| = \sum 1_{A_k} \|x\| =$$

$$= \sum \|1_{A_k} x\| = \sum \|1_{A_k} x_k\| = \sum 1_{A_k} \|x_k\| \leq \varepsilon.$$

This proves that $x \in U_{F,\varepsilon}$.

2 $\Rightarrow$ 1: Let $\mathscr{U}$ be a neighborhood base of $0 \in E$ for which each $U \in \mathscr{U}$ is L^0-convex, L^0-absorbing, L^0-balanced and stable. Consider the family of L^0-seminorms $\{\|\cdot\|_U : U \in \mathscr{U}\}$ and show that it induces the topology $\mathscr{T}$.

Given $U \in \mathscr{U}$, it is clear that $U \subset U_{\{\|\cdot\|_U\},1}$. Therefore, for $\varepsilon \in L^0_{++}$ there exists $V \in \mathscr{U}$ such that $\frac{1}{\varepsilon} V \subset U \subset U_{\{\|\cdot\|_U\},1}$. Thus, $V \subset U_{\{\|\cdot\|_U\},\varepsilon}$. On the other hand, for $U \in \mathscr{U}$, it holds that $U_{\{\|\cdot\|_U\},\frac{1}{2}} \subset \{x \in E \colon \|x\|_U < 1\} \subset U$.

2 $\Rightarrow$ 3 is obvious.

3 $\Rightarrow$ 2: Let $\mathscr{U}$ be a neighborhood base of $0 \in E$ consisting of stable and L^0-convex subsets of E. Consider the collection $\mathscr{V} := \{-U \cap U \colon U \in \mathscr{U}\}$. For any $U \in \mathscr{U}$, by continuity of the scalar product, there exists $V \in \mathscr{U}$ such that $V \subset -U$, hence there exists $W \in \mathscr{U}$ such that $W \subset U \cap V \subset -U \cap U$. This proves that $\mathscr{V}$ is a neighborhood base of $0 \in E$ for $\mathscr{T}$. Inspection shows that any $V \in \mathscr{V}$ is stable and L^0-convex. Fix $V := -U \cap U \in \mathscr{V}$. Given $\eta \in L^0$ with $|\eta| \leq 1$, set $a^+ := \{\eta \geq 0\}$ and $a^- := \{\eta < 0\}$. Then, using that V is L^0-convex we have

$$1_{a^+} \eta x = 1_{a^+}(\eta^+ x + (1 - \eta^+)0) \in 1_{a^+} V,$$

$$1_{a^-} \eta x = 1_{a^-}(\eta^-(-x) + (1 - \eta^-)0) \in 1_{a^-} V.$$

Since V is stable, we get that $\eta x \in V$. Since $x \in V$ is arbitrary, we obtain that $\eta V \subset V$. We conclude that V is L^0-balanced. Finally, by the continuity of the scalar product, it can be verified that V is L^0-absorbing. □

Theorem 2.4 in [36] claims that any locally L^0-convex topology is induced by a family of L^0-seminorms. The statement of Theorem 1.2.1 differs from Theorem 2.4 of [36] in requiring an extra condition over the elements of the neighborhood base of $0 \in E$; namely, being stable. Below, we provide an example of a locally L^0-convex topology which is not induced by any family of L^0-seminorms. This shows that this condition was unfortunately dropped in Theorem 2.4 of [36].

Example 1.2.1. *Let $(\Omega, \mathcal{F}, P)$ be a probability space such that Ω is not the union of finitely many atoms. (for example, $\Omega = (0,1)$, $\mathcal{F} = \mathcal{B}(\Omega)$ the Borel σ-algebra and $P := \lambda$ the Lebesgue measure). Let $(a_k) \in p(I)$ be infinite with $a_k > 0$ for all $k \in \mathbb{N}$.*

For $\varepsilon \in L^0_{++}$, we define the set

$$U_\varepsilon := \left\{\eta \in L^0 \colon \exists\, I \subset \mathbb{N} \text{ finite},\ |\eta 1_{a_i}| \leq \varepsilon\, \forall\, i \in \mathbb{N} - I\right\}.$$

Then, it is simple to verify that U_ε is L^0-convex, L^0-absorbing and L^0-balanced, and $\mathscr{U} := \left\{U_\varepsilon \colon \varepsilon \in L^0_{++}\right\}$ is a neighborhood base of $0 \in E$ which generates a locally L^0-convex topology $\mathscr{T}$.

Furthermore, note that $\varepsilon + 1 \notin U_\varepsilon$, $\varepsilon + 1 = \sum_k 1_{a_k}(\varepsilon + 1)$ and $1_{a_k}(\varepsilon + 1) \in U_\varepsilon$ for all $k \in \mathbb{N}$. This shows that U_ε is not stable.

Moreover, we claim that, for any other neighborhood base $\mathscr{V}$ of $0 \in E$ generating the same topology, there exists some $V \in \mathscr{V}$ which is not stable in L^0. Indeed, let us choose $V \in \mathscr{V}$ and

$\varepsilon \in L^0_{++}$ with $\varepsilon U_1 \subset V \subset U_1$. Clearly, $1_{a_k}\frac{2}{\varepsilon} \in 1_{a_k}U_1$ for each $k \in \mathbb{N}$. Consequently, since $\varepsilon U_1 \subset V$, we have that $1_{a_k}2 \in 1_{a_k}V$ for each $k \in \mathbb{N}$. However, $2 \notin V$ because $V \subset U_1$ and $2 \notin U_1$. We conclude that V is not stable.

Therefore, Theorem 1.2.1 tells us that $\mathscr{T}$ cannot be induced by any family of L^0-seminorms.

Finally, we have that $\left\{x \in L^0 \colon \|x\|_{U_\varepsilon} < 1\right\} \not\subseteq U_\varepsilon$ for some $\varepsilon \in L^0_{++}$. Otherwise, the family of L^0-seminorms $\{\|\cdot\|_{U_\varepsilon} \colon \varepsilon \in L^0_{++}\}$ would induce the topology.

In fact, we claim that $\|x\|_U = 0$ for all $x \in L^0$ and $U \in \mathscr{U}$. Since $\|\cdot\|_{U_1}$ is an L^0-seminorm, it suffices to show that $\|1\|_{U_1} = 0$. Indeed, to reach a contradiction, assume $\{\|1\|_{U_1} > 0\} > 0$. If so, then there exists $m \in \mathbb{N}$ such that $\{\|1\|_{U_1} > 0\} \wedge a_m > 0$. Let $a := \{\|1\|_{U_1} > 0\} \wedge a_m$, $\nu := \frac{1}{2}\|1\|_{U_1} + 1_{a^c}$, $\eta := 1_{a^c} + \nu 1_a$ and $x := 1_{a^c} + \frac{1}{\nu}1_a$. Then, we have that $1 = \eta x \in \eta U_1$ and $\{\|1\|_{U_1} > \eta\} > 0$, which is a contradiction.

In view of Theorem 1.2.1 and Example 1.2.1, we propose a new definition of locally L^0-convex module which we will adopt in the remainder of thesis:

Definition 1.2.1. *A topological L^0-module $(E, \mathscr{T})$ is called a* locally L^0-convex module *if there exists a neighborhood base $\mathscr{U}$ of $0 \in E$ such that any $U \in \mathscr{U}$ is stable, L^0-convex, L^0-absorbing and L^0-balanced. In this case, $\mathscr{T}$ is a locally L^0-convex topology.*

With Definition 1.2.1, and in view of Theorem 1.2.1, it is satisfied that any locally L^0-convex topology is induced by a family of L^0-seminorms.

Lemma 1.2.2. *Let $(E, \mathscr{T})$ be a topological L^0-module, $\mathscr{U}$ a neighborhood base of $0 \in E$ and S a subset of E. Then*

$$\overline{S} = \bigcap_{U \in \mathscr{U}} (S + U).$$

Proof. We can suppose w.l.o.g. that $\mathscr{U}$ contains all the neighborhoods of $0 \in E$. $x \in \overline{S}$ if and only if $(x + U) \cap S \neq \emptyset$ for all $U \in \mathscr{U}$, and the latter happens if and only if $x \in S - U$ for all $U \in \mathscr{U}$. Since U is a neighborhood of 0 if and only if $-U$ is a neighborhood of 0, the conclusion follows. □

Proposition 1.2.2. *Let $(E, \mathscr{T})$ be a locally L^0-convex module. The following are equivalent:*

1. *$\mathscr{T}$ is Hausdorff;*
2. *$\mathscr{T}$ is induced by a separated family of L^0-seminorms;*
3. *There exists a neighborhood base $\mathscr{U}$ of $0 \in E$ such that $\bigcap \mathscr{U} = \{0\}$;*
4. *$\overline{\{0\}} = \{0\}$.*

Proof. $1 \Rightarrow 3$: Let $\mathscr{U}$ be a neighborhood base of $0 \in E$ and take $x \in \bigcap \mathscr{U}$. Then $x = 0$; otherwise, we can find $U \in \mathscr{U}$ such that $x \notin U$, and then $x \notin \bigcap \mathscr{U}$.

$3 \Rightarrow 2$: Take a neighborhood base $\mathscr{U}$ of $0 \in E$ consisting of stable, L^0-convex, L^0-absorbing and L^0-balanced neighborhoods. Necessarily $\bigcap \mathscr{U} = \{0\}$. In the proof of Theorem 1.2.1 we saw that $\{\|\cdot\|_U \colon U \in \mathscr{U}\}$ induces $\mathscr{T}$. Suppose that $\sup_{U \in \mathscr{U}} \|x\|_U = 0$. Proposition 1.2.1 shows that $\{y \colon \|y\|_U < 1\} \subset U$ for all $U \in \mathscr{U}$. Thus $x \in \bigcap_U \{y \colon \|y\|_U < 1\} \subset \bigcap \mathscr{U} = \{0\}$. This proves that $x = 0$. Therefore, $\{\|\cdot\|_U \colon U \in \mathscr{U}\}$ is separated.

$2 \Rightarrow 1$: If $x \neq y$, then $z := x - y \neq 0$. Therefore, since $\mathscr{P}$ is separated, $\|z\| \neq 0$ for some L^0-seminorm $\|\cdot\| \in \mathscr{P}$. Let $a := \{\|z\| > 0\}$, $\varepsilon := 1_a\frac{1}{4}\|z\| + 1_{a^c}$ and $U := U_{\{\|\cdot\|\},\varepsilon}$. Then it is simple to verify that $(x + U) \cap (y + U) = \emptyset$.

$3 \Leftrightarrow 4$ is an immediate consequence of Lemma 1.2.1. □

In Example 1.1.2, we studied an L^0-module for which the uniqueness of concatenations fails. The following result shows that this situation is not possible in locally L^0-convex modules which are Hausdorff:

Proposition 1.2.3. *Let $(E, \mathscr{T})$ be a Hausdorff locally L^0-convex module. Then for given $(a_k) \in p(I)$ and $(x_k) \subset E$, there exists at most one countable concatenation of (x_k) along (a_k).*

Proof. Due to Proposition 1.2.2, we can find a neighborhood base $\mathscr{U}$ of $0 \in E$ such that $\bigcap \mathscr{U} = \{0\}$ and every $U \in \mathscr{U}$ is stable.

Suppose that $(a_k) \in p(I)$ and $(x_k) \subset E$. Let $x, y \in E$ be concatenation of (x_k) along (a_k), that is,

$$1_{a_k} x = 1_{a_k} y = 1_{a_k} x_k \quad \text{for all } k \in \mathbb{N}.$$

We have that $1_{a_k}(x - y) = 0$ for all $k \in \mathbb{N}$.[1] Fixed $U \in \mathscr{U}$, one has that $1_{a_k}(x - y) = 0 \in U$ for all $k \in \mathbb{N}$. Since U is stable, we have that $x - y \in U$.

Being $U \in \mathscr{U}$ arbitrary, it follows that $x - y \in \bigcap \mathscr{U} = \{0\}$. We conclude that $x = y$. □

Comments 1.2.1. *The failure of Theorem 2.4 in [36] was earlier conjectured by Guo et al. [54], however no counterexample was provided. Theorem 1.2.1 and Example 1.2.1 were independently obtained by Wu and Guo [112].*

However, Theorem 1.2.1 and Example 1.2.1 appeared first time in the earliest version of our publication [117] before than [112].

1.3 A Mazur theorem for L^0-normed modules with additive class of stable subsets

A well-known result of classical convex analysis is the Mazur theorem, which asserts that in a locally convex space the closure and the weak closure of a convex set coincide. We will find that a module analogue of the Mazur theorem for L^0-normed modules can be established without the necessity of the countable concatenation property. Namely, we will find that such a statement holds in the larger class of L^0-normed modules with additive class of stable subsets (see Definition 1.1.1).

Before stating the main result, we need some preliminary results. We will start by recalling the extension theorem for L^0-modules proven in [36], which does not need the countable concatenation property:

Lemma 1.3.1. *[36, Theorem 2.14] Suppose that $\|\cdot\| : E \to L^0$ is an L^0-seminorm, F is an L^0-submodule of E, and $\mu : E \to L^0$ an L^0-module homomorphism such that*

$$\mu(x) \leq \|x\| \quad \text{for all } x \in F.$$

Then μ extends to an L^0-module homomorphism $\bar{\mu} : E \to L^0$ such that $\bar{\mu}(x) \leq \|x\|$ for all $x \in E$.

Remark 1.3.1. *Hahn–Banach type extension theorems for modules appeared already in the fifties. These types of results started with [46], where modules over totally ordered rings were considered. Modules over rings which are algebraically and topologically isomorphic to the space of essentially bounded measurable functions on a finite measure space were studied later in [60, 109, 85]. Nowadays, it is well known that a Hahn–Banach type extension theorem for modules over more general ordered rings can be established, cf. [21, 111], which covers Lemma 1.3.1.*

[1] Recall that the condition $1_{a_k}(x - y) = 0$ for all $k \in \mathbb{N}$ does not guaranty that $x = y$ as shown in Example 1.1.2.

Lemma 1.3.2. *Let $(E, \mathscr{T})$ be a topological L^0-module and let $S \subset E$ be L^0-convex and L^0-absorbing. Then the following are equivalent:*

1. *$\|\cdot\|_S : E \to L^0$ is continuous;*
2. *$0 \in \overset{o}{S}$.*

In this case, if in addition S is stable, it is satisfied

$$\overline{S} = \{x \in E \colon \|x\|_S \leq 1\}.$$

Proof. The proof is word by word the same as the real case. As for the equality,

"$\subset$" follows from the continuity of $\|\cdot\|_S$.

"$\supset$": Let $x \in E$ be satisfying $\|x\|_S \leq 1$. By Lemma 1.2.1, one has that for any $\varepsilon \in L^0_{++}$ we can pick $\eta_\varepsilon \in L^0_{++}$ so that $1 \leq \eta_\varepsilon < 1 + \varepsilon$ and $x \in \eta_\varepsilon S$. Therefore, $(\eta_\varepsilon^{-1} x)_{\varepsilon \in L^0_{++}}$ is a net (viewing L^0_{++} directed downwards) in S, which converges to x, hence $x \in \overline{S}$. □

The example below shows that, for the equality proved in the latter proposition, S needs to be stable.

Example 1.3.1. *As in Example 1.2.1, let $(\Omega, \mathcal{F}, P)$ be a probability space such that Ω is not the union of finitely many atoms. Let $(a_k) \in p(I)$ be infinite with $a_k > 0$ for all $k \in \mathbb{N}$.*

For $\varepsilon \in L^0_{++}$, set

$$U_\varepsilon = \left\{\eta \in L^0 \colon \exists\, I \subset \mathbb{N} \text{ finite, } |\eta 1_{A_i}| \leq \varepsilon \,\forall\, i \in \mathbb{N} - I\right\}.$$

We saw in Example 1.2.1 that the collection $\{U_\varepsilon \colon \varepsilon \in L^0_{++}\}$ is a neighborhood base of $0 \in E$ of a locally L^0-convex topology. Besides, U_1 is L^0-convex, L^0-absorbing and closed. However, we know that $\|\cdot\|_{U_1} \equiv 0$, hence $\{x \in E \colon \|x\|_{U_1} \leq 1\} = E$.

Definition 1.3.1. *Suppose that S is a non-empty set of E, we define its* stable L^0-convex hull *by*

$$\mathrm{co}^{\mathfrak{s}}_{L^0}(S) := \bigcap \{U \subset E \colon U \text{ is } L^0\text{-convex and stable and } S \subset U\}.$$

Notice that $\mathrm{co}^{\mathfrak{s}}_{L^0}(S)$ is stable and L^0-convex and, moreover, it is the least set with these properties that contains S.

Finally, we have the following:

Theorem 1.3.1. *Let $(E, \|\cdot\|)$ be an L^0-normed module with additive class of stable subsets. Suppose that $(x_\gamma)_{\gamma \in \Gamma}$ is a net in E which converges to x w.r.t. $\sigma(E, E^*)$. Then, for each $\varepsilon \in L^0_{++}$ there exists*

$$z_\varepsilon \in \mathrm{co}^{\mathfrak{s}}_{L^0}\{x_\gamma \colon \gamma \in \Gamma\} \quad \textit{such that } \|x - z_\varepsilon\| \leq \varepsilon.$$

Proof. Let $S_1 := \mathrm{co}^{\mathfrak{s}}_{L^0}\{x_\gamma \colon \gamma \in \Gamma\}$. We may assume $0 \in S_1$, by replacing x by $x - x_{\gamma_0}$ and x_γ by $x_\gamma - x_{\gamma_0}$ for some fixed $\gamma_0 \in \Gamma$ and for all $\gamma \in \Gamma$.

To reach a contradiction, suppose that for all $z \in S_1$ there exists $a_z \in \mathcal{A} \setminus \{0\}$ such that

$$\|x - z\| > \varepsilon \quad \text{on } a_z. \tag{1.2}$$

Let $B_{\varepsilon/2}(0) := \{x \in E \colon \|x\| \leq \frac{\varepsilon}{2}\}$, and define

$$S := \bigcup_{z \in S_1} (z + B_{\varepsilon/2}(0)) = S_1 + B_{\varepsilon/2}(0).$$

Then S is an L^0-convex L^0-absorbing neighborhood of $0 \in E$, which is also stable as S_1 and $B_{\varepsilon/2}(0)$ are stable and E has additive class of stable subsets. Further, in view of (1.2), we have that for any $z \in S$, there exists $b_z \in \mathcal{A} \setminus \{0\}$ such that

$$\|x - z\| \geq \frac{\varepsilon}{2} \quad \text{on } b_z.$$

This implies that $x \notin \overline{S}$. Thus, due to Proposition 1.3.2, there exists $a \in \mathcal{A} \setminus \{0\}$ such that

$$\|x\|_S > 1 \quad \text{on } a. \tag{1.3}$$

We will prove that $\eta = \xi$ on a whenever $1_a \eta x = 1_a \xi x$. Indeed, suppose that $1_a \eta x = 1_a \xi x$ and set $b := \{\eta - \xi \geq 0\}$. Then

$$1_a|\eta - \xi|\|x\|_S \leq \|1_a|\eta - \xi|x\|_S = \|(1_b - 1_{b^c})1_a(\eta - \xi)x\|_S = \|0\|_S = 0.$$

Bearing in mind (1.3), we conclude that $\eta = \xi$ on a.

Then the mapping

$$\mu : \operatorname{span}_{L^0}\{x\} \longrightarrow L^0 \quad \mu(\eta x) = \eta 1_a \|x\|_S$$

is well-defined and is an L^0-module homomorphism. In addition, one has that

$$\mu(z) \leq \|z\|_S \quad \text{for all } z \in \operatorname{span}_{L^0}\{x\}.$$

Thus, by Lemma 1.3.1, μ extends to an L^0-module homomorphism $\bar{\mu}$ defined on E such that

$$\bar{\mu}(z) \leq \|z\|_S \quad \text{for all } z \in E.$$

Since S is a neighborhood of $0 \in E$, by Proposition 1.3.2, $\|\cdot\|_S$ is continuous. Hence $\bar{\mu}$ is continuous. Furthermore, we have

$$\sup_{z \in S_1} \bar{\mu}(z) \leq \sup_{z \in S} \bar{\mu}(z) \leq \sup_{z \in S} \|z\|_S \leq 1 < \|x\|_S = \bar{\mu}(x) \quad \text{on } a.$$

Therefore, x cannot be a $\sigma(E, E^*)$-accumulation point of S_1 contrary to the hypothesis of $(x_\gamma)_{\gamma \in \Gamma}$ converging to x. □

Finally, as a consequence of the result above, we obtain the announced module analogue of the Mazur theorem:

Corollary 1.3.1. *Let $(E, \|\cdot\|)$ be an L^0-normed module with additive class of stable subsets and let $S \subset E$ be stable and L^0-convex. Then we have*

$$\overline{S}^{\|\cdot\|} = \overline{S}^{\sigma(E,E^*)}.$$

Proof. The L^0-norm topology is finer than the weak topology; hence $\overline{S}^{\|\cdot\|} \subset \overline{S}^{\sigma(E,E^*)}$.

On the other hand, for given $x \in \overline{S}^{\sigma(E,E^*)}$, take a net $(x_\gamma)_{\gamma \in \Gamma}$ which converges to x w.r.t. $\sigma(E, E^*)$. Due to Theorem 1.3.1, for every $\varepsilon \in L^0_{++}$, we can find

$$z_\varepsilon \in \operatorname{co}^{\mathfrak{s}}_{L^0}\{x_\gamma : \gamma \in \Gamma\} \subset S \quad \text{such that } \|z_\varepsilon - x\| \leq \varepsilon.$$

This implies that $x \in \overline{S}^{\|\cdot\|}$. □

1.4 Locally stable L^0-convex modules

Henceforth, for simplicity in the exposition, we will assume always the countable concatenation property. Thus, in general, by an L^0-module we will mean an L^0-module with the countable concatenation property. In this case, as observed before, a non-empty subset S of E is stable if for every partition $(a_k) \in p(I)$ and sequence $(x_k) \subset E$ it holds that $\sum 1_{a_k} x_k \in S$.

Next, we introduce a similar notion for collections of subsets of E:

Definition 1.4.1. *A non-empty collection $\mathscr{C}$ of non-empty subsets of E is* stable *if for any countable family $(S_k) \subset \mathscr{C}$ and partition $(a_k) \in p(I)$ one has*

$$\sum 1_{a_k} S_k := \left\{ \sum 1_{a_k} x_k \colon x_k \in S_k \text{ for all } k \in \mathbb{N} \right\} \in \mathscr{C}.$$

Most of the main results of [36] are proven under the assumption that the locally L^0-convex topology is induced by a family of L^0-seminorms which is closed under the operations of finitely many suprema and countable concatenations. These conditions suggest the following:

Definition 1.4.2. *Let $\mathscr{P}$ be a family of L^0-seminorms on E. Given $(a_k) \in p(I)$, a family (F_k) of non-empty finite subsets of $\mathscr{P}$, and $\varepsilon \in L^0_{++}$, we define*

$$U_{(F_k),(a_k),\varepsilon} := \{x \in E \colon \sum 1_{a_k} \sup_{\|\cdot\| \in F_k} \|x\| \le \varepsilon\}.$$

Then

$$\mathscr{U} := \left\{ U_{(F_k),(a_k),\varepsilon} \colon \varepsilon \in L^0_{++},\ F_k \subset \mathscr{P} \text{ finite for all } k \in \mathbb{N},\ (a_k) \in p(I) \right\}$$

is a neighborhood base of $0 \in E$ for some locally L^0-convex topology on E, denoted by $\langle \mathscr{P} \rangle_{\mathfrak{s}}$, which is finer than $\langle \mathscr{P} \rangle$. This topology will be referred to as the topology stably induced *by $\mathscr{P}$.*

Remark 1.4.1. *Note that if $\mathscr{P}$ stably induces a topology $\mathscr{T}$ on E, then $\mathscr{T}$ is also induced (in the non-stable sense) by the family of L^0-seminorms*

$$\mathscr{P}_{\mathfrak{s}} := \left\{ \sum_{k \in \mathbb{N}} 1_{a_k} \sup_{\|\cdot\| \in F_k} \|x\| \colon (a_k) \in p(I),\ F_k \subset \mathscr{P} \text{ is non-empty and finite.} \right\},$$

that is, $\langle \mathscr{P}_{\mathfrak{s}} \rangle = \langle \mathscr{P} \rangle_{\mathfrak{s}}$.

In what follows, we will provide a result which characterizes when a locally L^0-convex topology is stably induced by a family of L^0-seminorms. We have the following:

Theorem 1.4.1. *Let $(E, \mathscr{T})$ be a topological L^0-module. The following conditions are equivalent:*

1. *there exists a family of L^0-seminorms $\mathscr{P}$ such that $\mathscr{T} = \langle \mathscr{P} \rangle_{\mathfrak{s}}$;*
2. *there exists a neighborhood base $\mathscr{U}$ of $0 \in E$ which is a stable collection and each $U \in \mathscr{U}$ is stable, L^0-convex, L^0-absorbing, L^0-balanced;*
3. *there exists a neighborhood base $\mathscr{U}$ of $0 \in E$ which is a stable collection and each $U \in \mathscr{U}$ is stable and L^0-convex.*

In this case, the collection $\{O \in \mathscr{T} : O \neq \emptyset\}$ is stable.

Proof. $1 \Rightarrow 2$: If $\mathscr{T}$ is stably induced by a family of L^0-seminorms $\mathscr{P}$, then the collection

$$\mathscr{U} := \left\{U_{(F_k),(a_k),\varepsilon}\colon \varepsilon \in L^0_{++},\ F_k \subset \mathscr{P} \text{ finite for all } k \in \mathbb{N}, (a_k) \in p(I)\right\}$$

is a neighborhood base of $0 \in E$ with respect to the topology $\mathscr{T}$, which satisfies the conditions 1-3 in the statement.

$2 \Rightarrow 1$: Let $\mathscr{U}$ be a stable neighborhood base of $0 \in E$ consisting of stable, L^0-convex, L^0-absorbing and L^0-balanced sets. In particular, from Theorem 1.2.1, we know that $\mathscr{T}$ is induced by the family of L^0-seminorms $\{\|\cdot\|_U\colon U \in \mathscr{U}\}$, where $\|\cdot\|_U : E \to L^0_+$ is the gauge function. Let us show that, in fact, $\mathscr{T}$ is stably induced by that family.

Indeed, let us fix a partition $(a_k) \in p(I)$, a sequence (F_k) of non-empty finite subsets of $\mathscr{U}$ and $\varepsilon \in L^0_{++}$. For each $k \in \mathbb{N}$, let us choose $U_k \in \mathscr{U}$ with $U_k \subset U$ for all $U \in F_k$. Then, one has that

$$\left\{x \in E\colon \sum 1_{a_k} \sup_{U\in F_k} \|x\|_U \leq \varepsilon\right\} \supset \left\{x \in E\colon \sum 1_{a_k}\|x\|_{U_k} \leq \frac{\varepsilon}{2}\right\}$$
$$= \left\{x \in E\colon \|x\|_{\sum 1_{a_k}U_k} \leq \frac{\varepsilon}{2}\right\}.$$

Since $\sum 1_{a_k}U_k \in \mathscr{U}$, the assertion follows.

$2 \Rightarrow 3$ is obvious.

$3 \Rightarrow 2$: If $\mathscr{U}$ is a stable neighborhood base of $0 \in E$ consisting of stable and L^0-convex sets, we define $\mathscr{V} := \{-U \cap U\colon U \in \mathscr{U}\}$. As in the proof of Theorem 1.2.1, we have that $\mathscr{V}$ is a neighborhood base of $0 \in E$ any $V \in \mathscr{V}$ is stable, L^0-convex, L^0-absorbing and L^0-balanced. Besides, a close look shows that $\mathscr{V}$ is a stable collection.

Finally, prove that $\{O \in \mathscr{T}\colon O \neq \emptyset\}$ is stable. Indeed, given $(a_k) \in p(I)$ and a countable family of non-empty open sets (O_k). Show that $O := \sum 1_{a_k}O_k$ is open. Indeed, for a fixed $x \in O$, let (x_k) be so that $x_k \in O_k$ and $1_{a_k}x_k = 1_{a_k}x$ for all k. Then, for each k, we can choose $U_k \in \mathscr{U}$ with $x_k + U_k \subset O_k$. Therefore $x + \sum 1_{a_k}U_k \subset \sum 1_{a_k}O_k = O$. □

Definition 1.4.3. *A topological L^0-module $(E, \mathscr{T})$ in the conditions of Theorem 1.4.1 is called a* locally stable L^0-convex module*. In this case, we say that $\mathscr{T}$ is a* locally stable L^0-convex topology*.*

In the following example, we give a locally L^0-convex topology which is induced by a family of L^0-seminorms and yet it is not stably induced by any family of L^0-seminorms.

Example 1.4.1. *Let $(\Omega, \mathcal{F}, P)$ be an atomless probability space and $(a_k) \in p(I)$ with $a_k > 0$ for each $k \in \mathbb{N}$. Take the L^0-module $E := (L^0)^{\mathbb{N}}$. For each $k \in \mathbb{N}$, consider the function $\|\cdot\|_k : E \to L^0_+$, $\|(x_n)_n\|_k := |x_k|$. Then $\{\|\cdot\|_k\colon k \in \mathbb{N}\}$ is a family of L^0-seminorms which induces the product topology on $(L^0)^{\mathbb{N}}$. However, it is not stably induced by a family of L^0-seminorms. Indeed, let us define $O_1 := (0,1)\times(L^0)^{\mathbb{N}}$, and for each $n \in \mathbb{N}$, let $O_n := (L^0)^{n-1}\times(0,1)\times(L^0)^{\mathbb{N}}$. Then $\sum 1_{a_k}O_k = \prod_{k\in\mathbb{N}} 1_{a_k}(0,1) + 1_{a_k^c}L^0$ is not an open subset. In view of the second part of Theorem 1.4.1, the product topology cannot be stably induced by any family of L^0-seminorms.*

Suppose that $(E, \mathscr{T})$ is a locally L^0-convex module. Recall that the family $\{|\mu(\cdot)|\colon \mu \in E^*\}$ induces the weak topology $\sigma(E, E^*)$ (see Example 1.2.1).

We can also consider the topology stably induced by this family of L^0-seminorms, which is called the *stably weak* topology and is denoted by $\sigma_{\mathfrak{s}}(E, E^*)$.

Similarly, we can define the *stably weak-$*$* topology $\sigma_{\mathfrak{s}}(E^*, E)$.

The stably weak (resp. weak-$*$) topology is finer than the weak (resp. weak-$*$) topology. The following example shows that both are not necessarily equal:

Example 1.4.2. *Consider a probability space $(\Omega, \mathcal{E}, P)$ such that $\mathcal{F} \subset \mathcal{E}$ is a sub-σ-algebra. If (p, q) are Hölder conjugates (i.e. $\frac{1}{p} + \frac{1}{q}$) with $1 \leq p < \infty$, then the map $T : L^p_{\mathcal{F}}(\mathcal{E}) \to [L^q_{\mathcal{F}}(\mathcal{E})]^*$, $y \mapsto T_y$ defined by $T_y(x) := E_P[xy|\mathcal{F}]$ is an $L^0(\mathcal{F})$-isometric isomorphism of $L^0(\mathcal{F})$-normed modules (see [36] and [51, Theorem 4.5]).*

We can consider the weak topology, and the collection of sets

$$U_{F,\varepsilon} := \left\{ x \in L^p_{\mathcal{F}}(\mathcal{E}) \colon \sup_{y \in F} |E_P[xy|\mathcal{F}]| \leq \varepsilon \right\},$$

where F is a non-empty finite subset of $L^q_{\mathcal{F}}(\mathcal{E})$ and $\varepsilon \in L^0_{++}(\mathcal{F})$, constitutes a neighborhood base $0 \in L^p_{\mathcal{F}}(\mathcal{E})$ for the weak topology.

Similarly, we can consider stably weak topology, and the collection of sets

$$U_{(F_k),(a_k),\varepsilon} := \left\{ x \in L^p_{\mathcal{F}}(\mathcal{E}) \colon \sum 1_{a_k} \sup_{y \in F_k} |E_P[xy|\mathcal{F}]| \leq \varepsilon \right\},$$

where $(a_k) \in p(I)$, (F_k) is a countable family of non-empty finite subsets of $L^q_{\mathcal{F}}(\mathcal{E})$ and $\varepsilon \in L^0_{++}(\mathcal{F})$, forms a neighborhood base $0 \in L^p_{\mathcal{F}}(\mathcal{E})$ for the stably weak topology.

Let us consider $\Omega = (0, 1)$, $\mathcal{E} = \mathcal{B}(\Omega)$ the Borel σ-algebra, $A_k = [\frac{1}{2^k}, \frac{1}{2^{k-1}})$ with $k \in \mathbb{N}$, $\mathcal{F} := \sigma(\{A_k \colon k \in \mathbb{N}\})$ and $P := \lambda$ the Lebesgue measure.

Also, for each $k \in \mathbb{N}$, let us define $L^2_k := L^2(A_k)$ considering the trace of $\mathcal{E}$ on A_k and the conditional probability $P(\cdot|A_k)$. For each $x \in L^2_k$ we denote $\|x|A_k\|_2 := E_P[x^2|A_k]^{1/2}$.

In this case,

$$L^0(\mathcal{F}) = \left\{ \sum_{k \in \mathbb{N}} 1_{a_k} r_k \colon r_k \in \mathbb{R} \right\},$$

In addition, one has that

$$L^2_{\mathcal{F}}(\mathcal{E}) = \left\{ \sum_{k \in \mathbb{N}} 1_{a_k} x_k \colon x_k|_{A_k} \in L^2_k \right\}$$

and, for $x \in L^2_{\mathcal{F}}(\mathcal{E})$, we have $\|x|\mathcal{F}\|_2 = \sum_{k \in \mathbb{N}} 1_{a_k} \|x|A_k\|_2$.

For each k, let us choose a countable set $\{y^k_n \colon n \in \mathbb{N}\}$ with $y^k_n|_{A_k} = z^k_n$ where $(z^k_n)_{n \in \mathbb{N}}$ is a linearly independent family in of L^2_k.

Let

$$U := \sum_{k \in \mathbb{N}} 1_{a_k} U_{\{y^k_1, \ldots, y^k_k\},1}.$$

We have that U is a neighborhood of $0 \in L^2_{\mathcal{F}}(\mathcal{E})$ for the topology $\sigma_{\mathfrak{s}}(L^2_{\mathcal{F}}(\mathcal{E}), L^2_{\mathcal{F}}(\mathcal{E}))$, but not for the topology $\sigma(L^2_{\mathcal{F}}(\mathcal{E}), L^2_{\mathcal{F}}(\mathcal{E}))$. Indeed, to reach a contradiction, let us assume that there is a finite subset F of $L^2_{\mathcal{F}}(\mathcal{E})$ and $\varepsilon \in L^0_{++}(\mathcal{F})$, where $\varepsilon = \sum_{k \in \mathbb{N}} 1_{a_k} r_k$ with $r_k \in \mathbb{R}^+$, so that $U_{F,\varepsilon} \subset U$. Now, let us take $k := \#F + 1$.

Let $F_k := \{y|_{A_k} \colon y \in F\}$, then

$$U_{F_k, r_k} \subset U_{\{z^k_1 \ldots z^k_k\},1} \quad \text{in } L^2_k.$$

For each $y \in L^2_k$, let us denote $\mu_y(x) := E_P[xy|A_k]$. Then, it follows that

$$\bigcap_{y \in F} \ker(\mu_y) \subset \bigcap_{i=1}^{k} \ker(\mu_{y^k_i}).$$

But this is impossible because $\bigcap_{y \in F} \ker(\mu_y)$ is a vector subspace of L^2_k with codimension less than k which is included into a vector subspace with codimension k.

An identical argument to that used to show Proposition 1.2.2 proves the following:

Proposition 1.4.1. *Let $(E,\mathscr{T})$ be a locally stable L^0-convex module. The following are equivalent:*

1. *$\mathscr{T}$ is Hausdorff;*
2. *$\mathscr{T}$ is stably induced by a separated family of L^0-seminorms;*
3. *there exists a stable neighborhood base $\mathscr{U}$ of $0\in E$ such that $\bigcap\mathscr{U}=\{0\}$ and so that any $U\in\mathscr{U}$ is stable, L^0-convex, L^0-absorbing and L^0-balanced.*
4. *$\overline{\{0\}}=\{0\}$.*

Proposition 1.4.2. *Let $(E,\mathscr{T})$ be a locally stable L^0-convex module and S a stable subset of E. Then $\overline{S}$ and $\overset{o}{S}$ are stable subsets.*

Proof. Let $\mathscr{U}$ be a stable neighborhood base of $0\in E$ consisting of stable subsets. Due to Lemma 1.2.2, one has $\overline{S}=\bigcap_{U\in\mathscr{U}}(S+U)$. Now suppose that $(a_k)\in p(I)$ and $(x_k)\subset\overline{S}$. Set $x:=\sum 1_{a_k}x_k$. Then, since any $U\in\mathscr{U}$ is stable, one has that $x\in S+U$. Thus $x\in\bigcap_{U\in\mathscr{U}}(S+U)=\overline{S}$.

If $(a_k)\in p(I)$ and $(x_k)\subset\overset{o}{S}$, then, for any $k\in\mathbb{N}$, we can pick $U_k\in\mathscr{U}$ with $x_k+U\subset S$. Set $x:=\sum 1_{a_k}x_k$ and $U:=\sum 1_{a_k}U_k$. We have that $x+U\subset S$. Since $U\in\mathscr{U}$, we conclude that $x\in\overset{o}{S}$. □

The following result relates the stability on the topological structure of a topological L^0-module with the stability on the algebraic structure of its topological dual:

Proposition 1.4.3. *Let $(E,\mathscr{T})$ be a topological L^0-module. If $\mathscr{T}\setminus\{\emptyset\}$ is stable, then $E^*[\mathscr{T}]$ has the countable concatenation property.*

Proof. Suppose that (μ_k) is a countable family of continuous L^0-module homomorphisms from E to L^0 and $(a_k)\in p(I)$, then we can define $\mu:=\sum 1_{a_k}\mu_k$, which is an L^0-module homomorphism from E to L^0. Let us show that μ is continuous. It suffices to study the continuity at $0\in E$. Fixed $\varepsilon\in L^0_{++}$, for each $k\in\mathbb{N}$, there exists $O_k\in\mathscr{T}$ with $0\in O_k$ so that $\mu(O_k)\subset B_\varepsilon$. If we set $O:=\sum 1_{a_k}O_k$, it is an open neighborhood of $0\in E$ since $\mathscr{T}$ is stable. We obtain $\mu(O)\subset\sum 1_{a_k}\mu(O_k)\subset B_\varepsilon$. □

The following example shows a locally L^0-convex module whose dual L^0-module has not the countable concatenation property:

Example 1.4.3. *Suppose that Ω is not the union of finitely many atoms. Let $E=(L^0)^{\mathbb{N}}$ be the set of all sequences in L^0. We consider the family of L^0-seminorms $\{\|\cdot\|_n\colon n\in\mathbb{N}\}$, where $\|(x_n)_n\|_k:=|x_k|$, $(x_n)\in E$. We claim that*

$$E^*=\{(x_n)\in E\colon\exists m\in\mathbb{N}\text{ such that }x_n=0\text{ for all }n\geq m\},$$

Notice that the latter space does not have the countable concatenation property. Indeed, take an infinite partition $(a_k)\in p(I)$ and let $e_k\in E$ denote the sequence such that its k^{th}-coordinate is one and zero otherwise. Clearly, the sequence $\sum 1_{a_k}e_k$ cannot be in E^*.

Let us prove the equality above. Any $x:=(x_n)$ with $x_n=0$ for all $n\geq m$ defines via

$$\mu_x((y_n)):=\sum_{n\geq1}x_ny_n$$

a continuous L^0-module homomorphism.

Conversely, suppose that $\mu \in E^*$. By continuity, there exists $\delta \in L^0_{++}$ and $m \in \mathbb{N}$ such that $|\mu(x)| < 1$ whenever $x = (x_n)$ satisfies $|x_n| < \delta$ for all $n = 1, \ldots, m$. Then $|\mu(re_n)| = r|\mu(e_n)| \leq 1$ for every $n > m$ and $r \in \mathbb{R}^+$. This implies that $\mu(e_n) = 0$ whenever $n > m$. Now define $x = (x_n)$ by $x_n := \mu(e_n)$ for $n \leq m$ and $x_n := 0$ otherwise. Then we have $\mu_x = \mu$.

Recall that a topological space is *anti-compact* if every compact subset is finite. We will show that, apart from boundary cases, any locally stable L^0-convex module is anti-compact.

The support of an L^0-module E is defined by

$$\operatorname{supp}(E) := \bigwedge \{a \in \mathcal{A} \colon 1_{a^c} E = \{0\}\}.$$

Lemma 1.4.1. *Let (η_n) be a sequence in L^0_+ with $\eta_n \neq 0$ for all $n \in \mathbb{N}$. Suppose that $\{\sup_{n\in\mathbb{N}} \eta_n > 0\}$ is atomless. Then there exists $\varepsilon \in L^0_{++}$ such that $\bar{P}(\eta_n \geq \varepsilon) > 0$ for all $n \in \mathbb{N}$.*

Proof. For each $n \in \mathbb{N}$, let $a_n := \{\eta_n > 0\}$. Since $\eta_n \neq 0$, one has that $a_n > 0$ for all $n \in \mathbb{N}$. Let $b_1 := a_1$. Since $a_2 \leq \{\sup_n \eta_n > 0\}$ and the latter is atomless, we can choose $b_2 > 0$ with $b_2 \leq a_2$ and $0 < \bar{P}(b_2) < \frac{1}{2}\bar{P}(b_1)$. By induction, for each $n \in \mathbb{N}$, we can find $a_{n+1} > 0$ with $b_{n+1} \leq a_{n+1}$ and $0 < \bar{P}(b_{n+1}) < \frac{1}{2}\bar{P}(b_n)$. For fixed $n \in \mathbb{N}$, it thus holds

$$\bar{P}(b_k) < \frac{1}{2^{k-n}}\bar{P}(b_n) \quad \text{for all } k > n.$$

Now, for each $n \in \mathbb{N}$, put $c_n := b_n - \bigvee_{k>n} b_k$. Then,

$$\bar{P}(c_n) \geq \bar{P}(b_n) - \sum_{k>n} \bar{P}(b_k) > \bar{P}(b_n) - \sum_{k>n} \frac{1}{2^{k-n}}\bar{P}(b_n) = 0.$$

Note that $(c_n) \in p\left(\bigvee c_n\right)$. Define $\varepsilon = \sum_{n\in\mathbb{N}} 1_{c_n}\eta_n/2 + 1_{\wedge c_n^c}$. Then $\varepsilon \in L^0_{++}$, and we conclude

$$\bar{P}(\eta_n \geq \varepsilon) \geq \bar{P}(c_n) > 0 \quad \text{for all } n \in \mathbb{N}.$$

□

Theorem 1.4.2. *Let $(E, \mathscr{T})$ be a Hausdorff locally stable L^0-convex module with supp$(E) > 0$. Then E is anti-compact if and only if supp(E) is atomless.*

Proof. Since $\mathscr{T}$ is Hausdorff, due to Proposition 1.4.1, $\mathscr{T}$ is stably induced by a separated family $\mathscr{P}$ of L^0-seminorms. For the sake of contradiction, suppose that $K \subset E$ is an infinite compact set. Then there exists an injective sequence (x_n) in K which has a cluster point $x_0 \in K$. We assume w.l.o.g. that $x_0 \neq x_n$ for all $n \in \mathbb{N}$. Since $\mathscr{P}$ is separated, for every $n \in \mathbb{N}$ there is $\|\cdot\|_n \in \mathscr{P}$ such that $\|x_0 - x_n\|_n \neq 0$.

Now, define $\rho : E \times E \to L^0$ by

$$\rho(x, y) := \sum_{n\in\mathbb{N}} \frac{1}{2^n} \frac{\|x - y\|_n}{1 + \|x - y\|_n}.$$

Then $\rho(x_n, x_0) \neq 0$ for all $n \in \mathbb{N}$ and $\{\sup_n \rho(x_n, x_0) > 0\} \leq \operatorname{supp}(E)$. By Lemma 1.4.1, there is $\varepsilon \in L^0_{++}$ such that $\bar{P}(\rho(x_n, x_0) \geq \varepsilon) > 0$ for all $n \in \mathbb{N}$. Let

$$V_\varepsilon(x_0) := \{y \in E \colon \rho(x_0, y) < \varepsilon\},$$

and notice that $x_n \notin V_\varepsilon(x_0)$ for all $n \in \mathbb{N}$. If we showed that $V_\varepsilon(x_0)$ is a neighborhood of x_0, then we would obtain the desired contradiction. Let $r_k := \sum_{n>k} \frac{1}{2^n}$, each k. Put $a_0 := 0$ and $a_k := \{r_k < \varepsilon/2\} \wedge a_{k-1}^c$ for $k \in \mathbb{N}$. Then $(a_k) \in p(I)$ and

$$\sum_{k\in\mathbb{N}} 1_{a_k} \sum_{n>k} \frac{1}{2^n} \frac{\|x_0 - y\|_n}{1 + \|x_0 - y\|_n} \leq \sum_{k\in\mathbb{N}} 1_{a_k} r_k < \frac{\varepsilon}{2} \quad \text{for all } y \in E.$$

Now,

$$0 = \inf_{\delta\in L^0_{++}} \left(\sum_{k\in\mathbb{N}} 1_{a_k} \sum_{1\leq n\leq k} \frac{1}{2^n} \frac{\delta}{1+\delta} \right).$$

By Lemma 1.2.1, we can find $\delta_0 \in L^0_{++}$ such that

$$\sum_{k\in\mathbb{N}} 1_{a_k} \sum_{1\leq n\leq k} \frac{1}{2^n} \frac{\delta}{1+\delta} < \frac{\varepsilon}{2} \quad \text{for all } 0 < \delta \leq \delta_0.$$

Finally, for each k, let $F_k := \{\|\cdot\|_n : n \leq k\}$. Then it is simple to verify that $x_0 + U_{(F_k),(a_k),\delta} \subset V_\varepsilon(x_0)$.

As for the converse, suppose that there exists an atom $a \leq \mathrm{supp}(E)$. Thus, for all $\|\cdot\| \in \mathscr{P}$ and $x \in E$ there exists a non-negative $r_{x,\|\cdot\|} \in \mathbb{R}$ such that $\|1_a x\| = 1_a r_{x,\|\cdot\|}$. For each $\|\cdot\| \in \mathscr{P}$, we define $\|\cdot\|' : E \to \mathbb{R}$ with $\|x\|' := E_P[\|1_a x\|]$ for each $\|\cdot\| \in \mathscr{P}$. Since $\mathscr{P}$ is separated, there exists $x_0 \in E$ and $\|\cdot\|_0 \in \mathscr{P}$ such that $\|x_0\|' > 0$. Then $\mathscr{P}' := \{\|\cdot\|' \colon \|\cdot\| \in \mathscr{P}\}$ is a set of real-valued seminorms which induces on the 1-dimensional vector space $E_0 := \{1_a r x_0 \colon r \in L^0\}$ a locally convex Hausdorff topology. By the Heine-Borel theorem, the unit ball K of E_0 is infinite and compact. Since $\mathscr{P}$ and $\mathscr{P}'$ induce the same topology on E_0, K is also compact in E. $\square$

1.5 Relations among three types of L^0-module topologies

Next, we will compare three different types of L^0-module topologies. Indeed, apart from the topologies induced and stably induced by families of L^0-seminorms, there is a third kind of topology that can be induced by a family of L^0-seminorms. Namely, the topology of the convergence in probability with respect to an L^0-norm was introduced by Haydon et al. [61]. This topology was independently introduced by Guo [49] and generalized to the case of a family of L^0-seminorms in [57]. In general, this topology is called (ϵ, λ)-topology and is defined as follows:

Definition 1.5.1. *Let $\mathscr{P}$ be a family of L^0-seminorms on E. Set*

$$U_{F,\epsilon,\lambda} := \{x \in E \colon \bar{P}(\sup_{\|\cdot\|\in F} \|x\| < \epsilon) > 1 - \lambda\}$$

where $F \subset \mathscr{P}$ is finite and ϵ, λ are positive real numbers with $\lambda < 1$. Then

$$\mathscr{U} := \{U_{F,\epsilon,\lambda} \colon F \subset \mathscr{P} \text{ finite}, \ \lambda, \epsilon \in \mathbb{R}^+, \ \lambda < 1\}$$

is a neighborhood base of $0 \in E$ of a Hausdorff topology which we denote by $\langle\mathscr{P}\rangle_{\epsilon,\lambda}$ and is referred to as the (ϵ, λ)-topology induced by $\mathscr{P}$.

Note that, for a given family of L^0-seminorms $\mathscr{P}$ defined on E, one has

$$\langle\mathscr{P}\rangle_{\epsilon,\lambda} \subset \langle\mathscr{P}\rangle \subset \langle\mathscr{P}\rangle_{\mathfrak{s}}.$$

Proposition 1.5.1. *Let $\mathscr{P}$ be a family of L^0-seminorms on E. Then $\langle\mathscr{P}\rangle_{\epsilon,\lambda} = \langle\mathscr{P}_{\mathfrak{s}}\rangle_{\epsilon,\lambda}$.*

Proof. Since $\mathscr{P} \subset \mathscr{P}_{\mathfrak{s}}$, it holds that $\langle\mathscr{P}\rangle_{\epsilon,\lambda} \subset \langle\mathscr{P}_{\mathfrak{s}}\rangle_{\epsilon,\lambda}$.

Conversely, suppose that $(a_k) \in p(I)$ and that (F_k) is a countable family of non-empty finite subsets of $\mathscr{P}$. Put $\|\cdot\| := \sum_k 1_{a_k} \sup_{\|\cdot\|\in F_k} \|\cdot\|$. Choose $m \in \mathbb{N}$ large enough such that $\bar{P}(\bigvee_{k>m} a_k) < \lambda/2$. Then we have

$$
\begin{aligned}
U_{\{\|\cdot\|\},\epsilon,\lambda} &= \left\{x \in E\colon \bar{P}\left(\bigvee_{k\geq 1}\left\{\sup_{\|\cdot\|\in F_k}\|x\| < \epsilon\right\} \wedge a_k\right) > 1-\lambda\right\} \\
&\supset \left\{x \in E\colon \bar{P}\left(\bigvee_{k=1}^{m}\left\{\sup_{\|\cdot\|\in F_k}\|x\| < \epsilon\right\} \wedge a_k\right) > 1-\lambda\right\} \\
&\supset \left\{x \in E\colon \bar{P}\left(\left\{\sup_{\|\cdot\|\in \cup_{k=1}^m F_k}\|x\| < \epsilon\right\} \wedge \bigvee_{k=1}^{m} a_k\right) > 1-\lambda\right\} \\
&\supset \left\{x \in E\colon \bar{P}\left(\sup_{\|\cdot\|\in \cup_{k=1}^m F_k}\|x\| < \epsilon\right) > 1-\lambda + \bar{P}\left(\left\{\sup_{\|\cdot\|\in \cup_{k=1}^m F_k}\|x\| < \epsilon\right\} \wedge \bigvee_{k>m} a_k\right)\right\} \\
&\supset \left\{x \in E\colon \bar{P}\left(\sup_{\|\cdot\|\in \cup_{k=1}^m F_k}\|x\| < \epsilon\right) > 1-\lambda + \bar{P}\left(\bigvee_{k>m} a_k\right)\right\} \\
&= \left\{x \in E\colon \bar{P}\left(\sup_{\|\cdot\|\in \cup_{k=1}^m F_k}\|x\| < \epsilon\right) > 1-\frac{\lambda}{2}\right\},
\end{aligned}
$$

where the last set is a basic neighbourhood of $\langle\mathscr{P}\rangle_{\epsilon,\lambda}$. $\square$

Proposition 1.5.2. *Let $\mathscr{P}_1, \mathscr{P}_2$ be families of L^0-seminorms on E. If $\langle\mathscr{P}_1\rangle = \langle\mathscr{P}_2\rangle$, then $\langle\mathscr{P}_1\rangle_{\epsilon,\lambda} = \langle\mathscr{P}_2\rangle_{\epsilon,\lambda}$.*

Proof. For $\epsilon, \lambda \in \mathbb{R}_+$ with $\lambda < 1$ and finite $F \subset \mathscr{P}_1$, choose finite $F' \subset \mathscr{P}_2$ and $\delta \in L^0_{++}$ such that

$$\left\{x \in E\colon \sup_{\|\cdot\|\in F'}\|x\| < \delta\right\} \subset \left\{x \in E\colon \sup_{\|\cdot\|\in F}\|x\| < \epsilon\right\}.$$

Let $(a_k) \in p(I)$ and $(\epsilon_k) \subset \mathbb{R}$ with $0 < \sum_k 1_{a_k}\epsilon_k \leq \delta$. Choose $m \in \mathbb{N}$ large enough so that

$\bar{P}(\bigvee_{k>m} a_k) < \lambda/2$. Then one has

$$\begin{aligned}
&\left\{x \in E \colon \bar{P}\left(\sup_{\|\cdot\| \in F} \|x\| < \epsilon\right) > 1-\lambda\right\} \\
&\supset \left\{x \in E \colon \bar{P}\left(\bigvee_{k\in\mathbb{N}} \left\{\sup_{\|\cdot\|\in F'} \|x\| < \epsilon_k\right\} \wedge a_k\right) > 1-\lambda\right\} \\
&\supset \left\{x \in E \colon \bar{P}\left(\bigvee_{k=1}^{m} \left\{\sup_{\|\cdot\|\in F'} \|x\| < \epsilon_k\right\} \wedge a_k\right) > 1-\lambda\right\} \\
&\supset \left\{x \in E \colon \bar{P}\left(\left\{\sup_{\|\cdot\|\in F'} \|x\| < \min_{1\le k\le m} \epsilon_k\right\} \wedge \bigvee_{k=1}^{m} a_k\right) > 1-\lambda\right\} \\
&\supset \left\{x \in E \colon \bar{P}\left(\sup_{\|\cdot\|\in F'} \|x\| < \min_{1\le k\le m} \epsilon_k\right) > 1-\lambda+\bar{P}\left(\bigvee_{k>m} a_k\right)\right\} \\
&\supset \left\{x \in E \colon \bar{P}\left(\sup_{\|\cdot\|\in F'} \|x\| < \min_{1\le k\le m} \epsilon_k\right) > 1-\frac{\lambda}{2}\right\}.
\end{aligned}$$

This proves that $\langle \mathscr{P}_1\rangle_{\epsilon,\lambda} \subset \langle \mathscr{P}_2\rangle_{\epsilon,\lambda}$. By interchanging the roles of $\mathscr{P}_1$ and $\mathscr{P}_2$, it follows the claim. □

Given a locally L^0-convex module $(E, \mathscr{T})$, due to Theorem 1.2.1, there exists a family $\mathscr{P}$ of L^0-seminorms such that $\mathscr{T} = \langle \mathscr{P}\rangle$. Proposition 1.5.2 tells us that the topology $\langle \mathscr{P}\rangle_{\epsilon,\lambda}$ does not depend on the choice of the inducing family of L^0-seminorms $\mathscr{P}$.

On the other hand, one can verify that $\langle \mathscr{P}\rangle_{\mathfrak{s}}$ is the coarsest locally stable L^0-convex topology that is finer than $\mathscr{T}$. Therefore, the topology $\langle \mathscr{P}\rangle_{\mathfrak{s}}$ does not depend on the choice of the inducing family of L^0-seminorms $\mathscr{P}$. In addition, Proposition 1.5.2 shows that $\mathscr{P}_{\mathfrak{s}}$ does not produce a new (ϵ, λ)-topology.

We conclude that, regardless the choice of $\mathscr{P}$, a locally L^0-convex topology $\mathscr{T}$ uniquely defines a (ϵ, λ)-topology, let us say $\mathscr{T}_{\epsilon,\lambda}$, and uniquely defines a locally stable L^0-convex topology, let us a say $\mathscr{T}_{\mathfrak{s}}$, such that

$$\mathscr{T}_{\epsilon,\lambda} \subset \mathscr{T} \subset \mathscr{T}_{\mathfrak{s}}.$$

The following result was proven in [51, Theorem 3.12] for the three topologies induced by a fixed family of L^0-seminorms. In view of the discussion above we have the following:

Proposition 1.5.3. *Let $(E, \mathscr{T})$ be a Hausdorff locally L^0-convex module and let $S \subset E$ be stable. Then*

$$\overline{S} = \overline{S}^{\mathfrak{s}} = \overline{S}^{\epsilon,\lambda}.$$

Proof. Take a separated family of L^0-seminorms $\mathscr{P}$ such that $\mathscr{T} = \langle \mathscr{P}\rangle$. Applying [51, Theorem 3.12] to the stable set S, we have that

$$\overline{S} = \overline{S}^{\langle \mathscr{P}\rangle} = \overline{S}^{\langle \mathscr{P}\rangle_{\epsilon,\lambda}} = \overline{S}^{\epsilon,\lambda}.$$

Now, applying again [51, Theorem 3.12] but to the family of L^0-seminorms $\mathscr{P}_{\mathfrak{s}}$, we obtain

$$\overline{S}^{\mathfrak{s}} = \overline{S}^{\langle \mathscr{P}_{\mathfrak{s}}\rangle} = \overline{S}^{\langle \mathscr{P}_{\mathfrak{s}}\rangle_{\epsilon,\lambda}}.$$

Finally, it follows from Proposition 1.5.1 that

$$\overline{S}^{\langle \mathscr{P}_{\mathfrak{s}}\rangle_{\epsilon,\lambda}} = \overline{S}^{\langle \mathscr{P}\rangle_{\epsilon,\lambda}} = \overline{S}^{\epsilon,\lambda}.$$

□

Having three types of topologies, we can define three dual L^0-modules. Namely, as previously $E^* := E^*[\mathscr{T}]$ denotes the set of all $\mathscr{T}$-continuous L^0-module homomorphisms; we denote by $E^*_{\mathfrak{s}} := E^*[\mathscr{T}_{\mathfrak{s}}]$ the set of all $\mathscr{T}_{\mathfrak{s}}$-continuous L^0-module homomorphisms; and similarly, we denote by $E^*_{\epsilon,\lambda} := E^*[\mathscr{T}_{\epsilon,\lambda}]$ the set of all $\mathscr{T}_{\epsilon,\lambda}$-continuous L^0-module homomorphisms.

As an immediate consequence of [54, Theorem 3.7] we have:

Proposition 1.5.4. *Let $(E, \mathscr{T})$ be a Hausdorff locally L^0-convex module. Then*

$$\mathfrak{s}(E^*) = E^*_{\mathfrak{s}} = E^*_{\epsilon,\lambda}.$$

Example 1.5.1. *Suppose that Ω is not the union of finitely many atoms. Let $E := (L^0)^{\mathbb{N}}$. We consider the family of L^0-seminorms $\{\|\cdot\|_n : n \in \mathbb{N}\}$, where $\|(x_n)_n\|_k := |x_k|$, $(x_n) \in E$. We claim that*

$$E^* = \{(x_n) \in E \colon \exists m \in \mathbb{N} \text{ such that } x_n = 0 \text{ for all } n \geq m\},$$
$$E^*_{\epsilon,\lambda} = E^*_{\mathfrak{s}} := \{(x_n) \in E \colon \exists \mathfrak{m} \in L^0(\mathbb{N}) \text{ such that } \sum_k 1_{\{\mathfrak{n}=k\}} x_k = 0 \text{ for all } \mathfrak{n} \geq \mathfrak{m},\ \mathfrak{n} \in L^0(\mathbb{N})\}.$$

Notice that the latter space is the stable hull of the former. In Example 1.4.3 we showed the first equality, by Proposition 1.5.4, we obtain the second equality.

Given a $(E, \mathscr{T})$ a locally L^0-convex module, we can define several types of weak topologies. Namely, three weak topologies

$$\sigma(E, E^*),\ \sigma(E, E^*_{\mathfrak{s}}),\ \sigma(E, E^*_{\epsilon,\lambda}),$$

three stably weak topologies

$$\sigma_{\mathfrak{s}}(E, E^*),\ \sigma_{\mathfrak{s}}(E, E^*_{\mathfrak{s}}),\ \sigma_{\mathfrak{s}}(E, E^*_{\epsilon,\lambda}),$$

and three (ϵ, λ)-weak topologies

$$\sigma_{\epsilon,\lambda}(E, E^*),\ \sigma_{\epsilon,\lambda}(E, E^*_{\mathfrak{s}}),\ \sigma_{\epsilon,\lambda}(E, E^*_{\epsilon,\lambda}).$$

Actually, some of them are repeated. Indeed, by Proposition 1.5.4, in each line the second and third topology are the same. Since $E^*_{\mathfrak{s}} = \mathfrak{s}(E^*)$, due to Proposition 1.5.1, we get that the three topologies of the last line are the same. Also, clearly, $\sigma_{\mathfrak{s}}(E, E^*) = \sigma_{\mathfrak{s}}(E, E^*_{\mathfrak{s}})$. Then we have

$$\begin{aligned}\sigma_{\epsilon,\lambda}(E, E^*) =&\sigma_{\epsilon,\lambda}(E, E^*_{\mathfrak{s}}) = \sigma_{\epsilon,\lambda}(E, E^*_{\epsilon,\lambda}) \subset \sigma(E, E^*) \subset \sigma(E, E^*_{\mathfrak{s}}) = \sigma(E, E^*_{\epsilon,\lambda}) \\ &\subset \sigma_{\mathfrak{s}}(E, E^*) = \sigma_{\mathfrak{s}}(E, E^*_{\mathfrak{s}}) = \sigma_{\mathfrak{s}}(E, E^*_{\epsilon,\lambda}).\end{aligned}$$

Notice that, due to example 1.4.2, $\sigma(E, E^*_{\mathfrak{s}})$ and $\sigma_{\mathfrak{s}}(E, E^*)$ are not necessarily the same topology.

The next result is an immediate consequence of Corollary 1.3.1 taking into account the relations shown above. Notice that, since we are considering always E with the countable concatenation property, we do not need to assume that E has additive class of stable subsets as in the statement of Corollary 1.3.1:

Theorem 1.5.1. *Let $(E, \|\cdot\|)$ be an L^0-normed module and let $S \subset E$ be stable and L^0-convex. Then we have*

$$\begin{aligned}&\overline{S}^{\|\cdot\|} = \overline{S}^{\|\cdot\|_{\epsilon,\lambda}} \\ =&\overline{S}^{\sigma(E,E^*)} = \overline{S}^{\sigma(E,E^*_{\mathfrak{s}})} = \overline{S}^{\sigma(E,E^*_{\epsilon,\lambda})} \\ =&\overline{S}^{\sigma_{\mathfrak{s}}(E,E^*)} = \overline{S}^{\sigma_{\mathfrak{s}}(E,E^*_{\mathfrak{s}})} = \overline{S}^{\sigma_{\mathfrak{s}}(E,E^*_{\epsilon,\lambda})} \\ =&\overline{S}^{\sigma_{\epsilon,\lambda}(E,E^*)} = \overline{S}^{\sigma_{\epsilon,\lambda}(E,E^*_{\mathfrak{s}})} = \overline{S}^{\sigma_{\epsilon,\lambda}(E,E^*_{\epsilon,\lambda})}.\end{aligned}$$

Chapter 2

A Boolean-valued model approach to L^0-convex analysis

This chapter aims to establish a Boolean-valued transfer principle from classical convex analysis to L^0-convex analysis. In general, Boolean-valued models provide a technology for expanding the content of known theorems of set theory to other models of set theory, giving rise to a new interpretation of each theorem with new and often non-obvious content. On this basis, we will establish a method to interpret any theorem of convex analysis as a theorem of L^0-convex analysis.

In Section 2.1, we recall the construction of the Boolean-valued model $V^{(\mathcal{A})}$ associated to a complete Boolean algebra $\mathcal{A}$ and review well-known elements and principles of Boolean-valued models that we will use throughout this chapter.

In Section 2.2, for fixed an underlying probability space $(\Omega, \mathcal{F}, P)$ we consider the Boolean-valued model $V^{(\mathcal{A})}$ associated to the probability algebra $\mathcal{A} := \bar{\mathcal{F}}$. We study a precise connection between locally stable L^0-convex modules and locally convex spaces within the Boolean-valued model $V^{(\mathcal{A})}$. Specifically, we show that for any locally stable L^0-convex module it can be constructed a tailored locally convex space within $V^{(\mathcal{A})}$. We prove that this assignment is an equivalence between the category of locally stable L^0-convex modules and continuous L^0-module homomorphisms and the category of locally convex spaces and linear continuous functions within $V^{(\mathcal{A})}$.

In Section 2.3, we list different elements of L^0-convex analysis and, by means of suitable *ascent operations*, discuss their meaning within $V^{(\mathcal{A})}$, providing a pull of different objects that will used later to build up statements of module analogues of available convex analysis theorems, which will hold due to the transfer principle of Boolean-valued models.

Finally, Section 2.4 is devoted to instances of application of the Boolean-valued transfer principle established. In particular, we prove that the main results in [36] are consequences of the Boolean-valued transfer principle. The number of instances of application can be easily increased. Nevertheless, we provide several modules analogues of classical results: for instance, versions of the Brouwer's fixed point, the Eberlein-Šmulian theorem, the James' compactness theorem, the Krein-Šmulian theorem and the Mazur's theorem.

2.1 Foundations of Boolean-valued models

Let us try to give an intuitive idea of what Boolean-valued analysis is. We would like to talk about what will happen at a particular moment in the future. This future is uncertain, it is influenced by events that we do not know yet. These possible events that might influence the future will be coded by a complete Boolean algebra $\mathcal{A} = (\mathcal{A}, \vee, \wedge, \cdot^c, 0, I)$.

Let us see some particular examples. The simplest case that we can think of is that the future that we are interested in is completely determined by the result of flipping a coin. In that case, the algebra of events is $\mathcal{A} = \{a, a^c, 0, I\}$ where a is the event "we get head", and its negation a^c is "we get tail". In the algebra of events we also have all events that we can formulate combining others, and so $a \vee a^c = I$ is the event "we get either head or tail", which is just the true event, and $a \wedge a^c = 0$ is the event "we get both head and tail" which is just the false event. Another example is that our future depends on a randomly chosen (say with Gaussian probability) real number $\eta(\omega)$. In that case, the algebra of events would be the probability algebra of the Borel σ-algebra.

So let us fix the Boolean algebra $\mathcal{A}$ of all the events that we can talk about and influence the future. The next element of our theory are the *names*. The names are the nouns of the language, with which we talk about objects in the future, despite the uncertainties. In the flipping coin example, suppose that I have five dollars and I bet two dollars that the outcome will be tail. Then I can consider the name $\mathbb{x}$ that represents the amount of money that I will have in the future. The actual value of $\mathbb{x}$ is unknown, it could be 3 or 7 depending on the coin. In the very simple flipping coin case, our name could be denoted by $3a + 7a^c$, and we have names for different gamblings. In the random real case, names will look more complicated but the idea is similar. Examples of names would be $\mathbb{y}$ that would take value 1 if the random real is positive or -1 otherwise, which could be denoted by $-1\{\eta \leq 0\} + 1\{\eta > 0\}$, and also $\mathbb{r}$ the name for the random real itself.

A special kind of names are those which do not really depend on the unknown events, and those are represented by a ✓ symbol above. For example, $\check{5}$ is a name which represents the number 5, no matter what the coin did or what the random real actually happens to be.

Once we understand the idea of a name, the next step is formulating statements about names and deciding what are the truth values of such statements. Playing with the names given above in the flipping coin case, it makes sense to formulate the following statements: $P1$. $\mathbb{x}$ is a positive real number, $P2$. $\mathbb{x} = \check{7}$, $P3$. $\mathbb{x} < \check{4}$, $P4$. $\mathbb{x} = \mathbb{r}^2$. While $P1$ is clearly true and $P4$ is clearly false, for $P2$ and $P3$ we may say that *it depends on what the coin will do*. In Boolean-valued analysis, these statements are not assigned a binary truth value of true or false. The truth value of a sentence P, denoted by $[\![P]\!]$ is an element of the measure algebra $\mathcal{A}$ that corresponds to the event that describes when this sentence is true. Thus $[\![P2]\!] = a^c$ because I have 7 dollars if and only if the flipping will give a tail, and similarly $[\![P3]\!] = a$. In the random case, for instance, the truth value of the sentence $\check{2}\mathbb{r} < \check{4}$ is exactly the element $\{\eta < 2\}$ of the measure algebra, while $[\![\mathbb{r} > \mathbb{y}]\!]$ is precisely $\{-1 < \eta < 0\} \vee \{1 < \eta\}$. In all these examples, we are using names for real numbers, but the idea is more general, we can have names for functions, sets, Banach spaces or any mathematical object we want, and state any kind of properties we wish in formal mathematical language. In the framework of set theory, any mathematical object can be considered as a set and any mathematical statement can be re-stated in terms of the membership relation $\in$ between sets.

The precise formulation of Boolean-valued analysis requires some familiarity with the basics of set theory and logic, and in particular with first-order logic, ordinals and transfinite induction (for a review of basic set theory see Appendix A). However, if one understands the key ideas and principles, it is possible to work with Boolean-valued models avoiding the underlying machinery that can be conveniently hidden in a black box. Next, we make a review which is based on [81, Chapter 2], [66, Chapter 14], [107] and [10].

Let us consider a universe of sets V satisfying the axioms of the Zermelo-Fraenkel set theory with the axiom of choice (ZFC) (for a review of ZFC set theory see Section A.2), and the first-order language of set theory $\mathcal{L}$ which allows the formulation of statements about the elements of V (for a precise description of $\mathcal{L}$ see Section A.1). In the universe V we have all possible mathematical objects (real numbers, topological spaces, etc.) that we can talk about in a context of total certainty. The language $\mathcal{L}$ consists of nouns for the elements of V plus a finite list of logic symbols ($\forall$, $\wedge$, $\neg$ and parenthesis), variables and the predicates $=$ and $\in$. Though we usually use a much richer language by introducing more and more intricate definitions, in the end, any usual mathematical statement can be written using only those mentioned. The elements of the universe V are classified into a transfinite hierarchy: $V_0 \subset V_1 \subset V_2 \subset \dots V_\omega \subset V_{\omega+1} \subset \dots$, where $V_0 = \emptyset$, $V_{\alpha+1} = \mathcal{P}(V_\alpha)$ is the family of all sets whose elements come from V_α for an ordinal α, and $V_\beta = \bigcup_{\alpha<\beta} V_\alpha$ for limit ordinal β (for a detailed description of the cumulative hierarchy of set theory see Section A.5).

Now consider a complete Boolean algebra $\mathcal{A} = (\mathcal{A}, \vee, \wedge, \cdot^c, 0, I)$ which is an element of V, which we can think that models the different events in a future time $t > 0$. Next, we will construct $V^{(\mathcal{A})}$, the *Boolean-valued model* of $\mathcal{A}$, whose elements are called *$\mathcal{A}$-valued sets*, that we interpret as the objects which we can talk about in the future. We will define functions $\mathbb{x} : \mathrm{dom}(\mathbb{x}) \to \mathcal{A}$. The idea is that for $\mathbb{y} \in \mathrm{dom}(\mathbb{x})$, $\mathbb{y}$ will become an element of $\mathbb{x}$ in the future if $\mathbb{x}(\mathbb{y})$ happens.

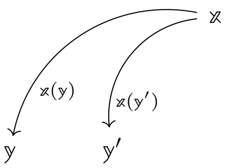

Figure 2.1: Boolean-valued membership relation in $\mathbb{x}$.

We proceed by transfinite induction over the class Ord of ordinals of the universe V (for a review of ordinals and transfinite induction see Section A.4). We start by defining

$$V_0^{(\mathcal{A})} := \emptyset.$$

If $\alpha + 1$ is the successor of the ordinal α, we define

$$V_{\alpha+1}^{(\mathcal{A})} := \left\{ \mathbb{x} \colon \mathbb{x} \text{ is a function } \mathbb{x} : \mathrm{dom}(\mathbb{x}) \to \mathcal{A} \text{ with } \mathrm{dom}(\mathbb{x}) \subset V_\alpha^{(\mathcal{A})} \right\}.$$

If α is a limit ordinal, we set $V_\alpha^{(\mathcal{A})} := \bigcup\limits_{\beta<\alpha} V_\beta^{(\mathcal{A})}$. Finally, let $V^{(\mathcal{A})} := \bigcup\limits_{\alpha\in\mathrm{Ord}} V_\alpha^{(\mathcal{A})}$.

Given an element $\mathbb{x}$ in $V^{(\mathcal{A})}$ we define its *rank*, denoted by $\mathrm{rank}(\mathbb{x})$, as the least ordinal α such that $\mathbb{x}$ is in $V_{\alpha+1}^{(\mathcal{A})}$. Notice that if $\mathbb{y} \in \mathrm{dom}(\mathbb{x})$, then $\mathrm{rank}(\mathbb{y}) < \mathrm{rank}(\mathbb{x})$.

We consider a first-order language which allows us to produce statements about $V^{(\mathcal{A})}$. Namely, let $\mathcal{L}^{(\mathcal{A})}$ be the first-order language which is the extension of $\mathcal{L}$ by adding *names* for denoting elements of $V^{(\mathcal{A})}$. Throughout, we will not distinguish between an $\mathcal{A}$-valued set and its name, so hereafter the elements of $V^{(\mathcal{A})}$ will be referred to as names.

Suppose that φ is any formula of the language $\mathcal{L}^{(\mathcal{A})}$, its *Boolean truth value* $[\![\varphi]\!] \in \mathcal{A}$ is defined by induction in the length of φ. We start by defining the Boolean truth value of the *atomic formulas* $\mathbb{x} \in \mathbb{y}$ and $\mathbb{x} = \mathbb{y}$ for $\mathbb{x}$ and $\mathbb{y}$ in $V^{(\mathcal{A})}$. Namely, proceeding by transfinite recursion on pairs $(\mathrm{rank}(\mathbb{x}), \mathrm{rank}(\mathbb{y}))$ with the canonical order of $\mathrm{Ord} \times \mathrm{Ord}$ (see Section A.4). We define

$$[\![\mathbb{x} \in \mathbb{y}]\!] = \bigvee_{\mathbb{t} \in \mathrm{dom}(\mathbb{y})} \mathbb{y}(\mathbb{t}) \wedge [\![\mathbb{t} = \mathbb{x}]\!],$$

$$[\![\mathbb{x} = \mathbb{y}]\!] = \bigwedge_{\mathbb{t} \in \mathrm{dom}(\mathbb{x})} (\mathbb{x}(\mathbb{t}) \Rightarrow [\![\mathbb{t} \in \mathbb{y}]\!]) \wedge \bigwedge_{\mathbb{t} \in \mathrm{dom}(\mathbb{y})} (\mathbb{y}(\mathbb{t}) \Rightarrow [\![\mathbb{t} \in \mathbb{x}]\!]),$$

where, for $a, b \in \mathcal{A}$, we denote $a \Rightarrow b := a^c \vee b$. For non-atomic formulas we have

$$[\![(\exists x)\varphi(x)]\!] := \bigvee_{\mathbb{u} \in V^{(\mathcal{A})}} [\![\varphi(\mathbb{u})]\!] \quad \text{and} \quad [\![(\forall x)\varphi(x)]\!] := \bigwedge_{\mathbb{u} \in V^{(\mathcal{A})}} [\![\varphi(\mathbb{u})]\!];$$

$$[\![\varphi \wedge \psi]\!] := [\![\varphi]\!] \wedge [\![\psi]\!] \quad \text{and} \quad [\![\neg\varphi]\!] := [\![\varphi]\!]^c.$$

Suppose that $\varphi(u_1, \ldots, u_n)$ is an n-ary formula ($n \geq 0$) in $\mathcal{L}$ with free variables $u_1, \ldots, u_n$. The above constructions give meaning to the formula $\varphi(\mathbb{u}_1, \ldots, \mathbb{u}_n)$ for members $\mathbb{u}_1, \ldots, \mathbb{u}_n$ of $V^{(\mathcal{A})}$. We say that the formula $\varphi(\mathbb{u}_1, \ldots, \mathbb{u}_n)$ is valid in $V^{(\mathcal{A})}$ if $[\![\varphi(\mathbb{u}_1, \ldots, \mathbb{u}_n)]\!] = I$.

The following properties are well-known (see eg [10, Theorem 1.17] or [81, Theorem 2.1.8]):

Proposition 2.1.1. *Let $\mathbb{x}, \mathbb{y}, \mathbb{z}$ be members of $V^{(\mathcal{A})}$. The following is satisfied:*

1. $[\![\mathbb{x} = \mathbb{x}]\!] = I$;
2. $\mathbb{x}(\mathbb{t}) \leq [\![\mathbb{t} \in \mathbb{x}]\!]$ *for all* $\mathbb{t} \in \mathrm{dom}(\mathbb{x})$;
3. $[\![\mathbb{x} = \mathbb{y}]\!] = [\![\mathbb{y} = \mathbb{x}]\!]$;
4. $[\![\mathbb{x} = \mathbb{y}]\!] \wedge [\![\mathbb{y} = \mathbb{z}]\!] \leq [\![\mathbb{x} = \mathbb{z}]\!]$;
5. $[\![\mathbb{x} = \mathbb{y}]\!] \wedge [\![\mathbb{z} \in \mathbb{x}]\!] \leq [\![\mathbb{z} \in \mathbb{y}]\!]$;
6. $[\![\mathbb{x} = \mathbb{y}]\!] \wedge [\![\mathbb{x} \in \mathbb{z}]\!] \leq [\![\mathbb{y} \in \mathbb{z}]\!]$;
7. *If $\varphi(u)$ is a formula with one free variable u, then* $[\![\mathbb{x} = \mathbb{y}]\!] \wedge [\![\varphi(\mathbb{x})]\!] \leq [\![\varphi(\mathbb{y})]\!]$.

We say that two names $\mathbb{x}, \mathbb{y}$ are equivalent when $[\![\mathbb{x} = \mathbb{y}]\!] = I$. Note that if $\varphi(u)$ is a formula with one free variable u, due to 6 of Proposition 2.1.1,

$$[\![\mathbb{x} = \mathbb{y}]\!] = I \quad \text{implies} \quad [\![\varphi(\mathbb{x})]\!] = [\![\varphi(\mathbb{y})]\!],$$

that is, the truth value of a formula is not affected when we change a name by an equivalent one. However, the relation $[\![\mathbb{x} = \mathbb{y}]\!] = I$ does not mean that the functions $\mathbb{x}$ and $\mathbb{y}$ (considered as elements of V) coincide. For example, the empty function $\mathbb{x} := \emptyset$ and the function $\mathbb{y} : \{\emptyset\} \to \mathcal{A}$, $\mathbb{y}(\emptyset) = 0$, are different as functions; however, $[\![\mathbb{x} = \mathbb{y}]\!] = I$. In order to avoid technical difficulties, we will construct the so-called *separated universe*. Namely, let $\overline{V}^{(\mathcal{A})}$ be the subclass of $V^{(\mathcal{A})}$ defined by choosing a representative in each class of equivalence $\{(\mathbb{x}, \mathbb{y}) \colon [\![\mathbb{x} = \mathbb{y}]\!] = I\}$. This can be done by transfinite recursion. Indeed, we set $\overline{V}_0^{(\mathcal{A})} := \emptyset$. If $\alpha + 1$ is a successor ordinal, then we choose a representative of each class of equivalence

$$\{(\mathbb{x}, \mathbb{y}) \colon \mathbb{x}, \mathbb{y} \in V_{\alpha+1}^{(\mathcal{A})} \text{ such that } [\![\mathbb{x} = \mathbb{y}]\!] = I,\ [\![\mathbb{x} = \mathbb{z}]\!] < I \text{ for all } \mathbb{z} \in \overline{V}_\alpha^{(\mathcal{A})}\},$$

and set $\overline{V}^{(\mathcal{A})}_{\alpha+1}$ the collection of all these representatives. If α is a limit ordinal, we set

$$\overline{V}^{(\mathcal{A})}_{\alpha} := \bigcup_{\beta<\alpha} \overline{V}^{(\mathcal{A})}_{\beta}.$$

Finally, let $\overline{V}^{(\mathcal{A})} := \bigcup_{\alpha\in\mathrm{Ord}} \overline{V}^{(\mathcal{A})}_{\alpha}$.

To avoid the mentioned difficulties, most of the time we will work with $\overline{V}^{(\mathcal{A})}$. Sometimes, we will construct specific names in $V^{(\mathcal{A})}$ by means of an explicit function, but just after this we will immediately take a canonical representative in $\overline{V}^{(\mathcal{A})}$.

The universe V can be embedded into $\overline{V}^{(\mathcal{A})}$ as follows: given x in V, we define its *canonical name* $\check{x}$ in $\overline{V}^{(\mathcal{A})}$ by transfinite induction. We set $\check{\emptyset} := \emptyset$ and for $x \in V$, we define $\check{x}$ to be the unique representative in $\overline{V}^{(\mathcal{A})}$ of the name given by the function

$$\{\check{y} \colon y \in x\} \longrightarrow \mathcal{A} \quad \check{y} \mapsto I.$$

Given a name $\mathbb{x}$ with $[\![\mathbb{x} \neq \emptyset]\!] = I$ we define its *descent* by

$$\mathbb{x}{\downarrow} := \{\mathbb{y} \in \overline{V}^{(\mathcal{A})} \colon [\![\mathbb{y} \in \mathbb{x}]\!] = I\}.$$

Notice that, if $\mathbb{x} \in V^{(\mathcal{A})}_{\alpha}$, then $\mathbb{x}{\downarrow} \subset V^{(\mathcal{A})}_{\alpha}$. Therefore, we have that $\mathbb{x}{\downarrow}$ is a set, that is, $\mathbb{x}{\downarrow}$ an element of V. Besides, due to the maximum principle, we have that $\mathbb{x}{\downarrow} \neq \emptyset$.

Now, suppose that $\mathbb{x}$ is a name with $[\![\mathbb{x} \neq \emptyset]\!] = I$ and such that any $\mathbb{y} \in \mathbb{x}{\downarrow}$ satisfies again $[\![\mathbb{y} \neq \emptyset]\!] = I$, then we define its *second descent* by

$$\mathbb{y}{\Downarrow} := \{\mathbb{z}{\downarrow} \colon \mathbb{z} \in \mathbb{y}{\downarrow}\}.$$

Similarly, it is clear that we could define the *third descent*, *fourth descent*, and so on. However, in this work, we will be only interested in the first and second descents.

Proposition 2.1.2. *[81, 3.2.3.(3)] Suppose that $\mathbb{x}$, $\mathbb{y}$ are names with $[\![(\mathbb{x} \neq \emptyset) \wedge (\mathbb{y} \neq \emptyset)]\!] = I$, then*

$$[\![\mathbb{x} = \mathbb{y}]\!] = I \quad \text{if and only if} \quad \mathbb{x}{\downarrow} = \mathbb{y}{\downarrow}.$$

Next, we will list some well-known principles which allow us to manipulate the Boolean-valued model $V^{(\mathcal{A})}$ (see eg [10, 81]).

The axioms of ZFC (see Section A.2) are valid in $V^{(\mathcal{A})}$, i.e. for any axiom Ax_i, $i = 1, \ldots, 9$, we have that $[\![Ax_i]\!] = I$ for all $i = 1, \ldots, 9$. Further, the usual logic rules of inference preserve the Boolean truth values of formulas. This amounts to the following:

Theorem 2.1.1. *(Transfer Principle) If φ is a theorem of ZFC, then $[\![\varphi]\!] = I$.*

The transfer principle will be crucial in this thesis as it will allow us to apply any known theorem of ZFC to our discussion.

Theorem 2.1.2. *(Maximum Principle) Let $\varphi(u)$ be a formula with one free variable u. Then there exists a name $\mathbb{u}$ such that $[\![\varphi(\mathbb{u})]\!] = [\![\exists x \varphi(x)]\!]$.*

Theorem 2.1.3. *(Mixing Principle) Let $(a_i)_{i\in\Delta} \in p(I)$ and let $(\mathbb{x}_i)$ be a family of names. Then there exists a name $\mathbb{x}$ such that $[\![\mathbb{x} = \mathbb{x}_i]\!] \geq a_i$ for all $i \in \Delta$. Moreover, if $\mathbb{y}$ is another name which satisfies $[\![\mathbb{y} = \mathbb{x}_i]\!] \geq a_i$ for all $i \in \Delta$, then $[\![\mathbb{x} = \mathbb{y}]\!] = I$.*

Given $(a_i) \in p(I)$ and a family of names $(\mathbb{x}_i)$, we denote by $\sum \mathbb{x}_i a_i$ the unique member $\mathbb{x}$ of $\overline{V}^{(\mathcal{A})}$ satisfying $[\![\mathbb{x} = \mathbb{x}_i]\!] \geq a_i$ for all i.

If $\mathbb{x}$ is a name with $[\![\mathbb{x} \neq \emptyset]\!] = I$, $(\mathbb{u}_i)$ is a family in $\mathbb{x}\!\downarrow$ and $(a_i) \in p(I)$, we can consider $\mathbb{u} := \sum \mathbb{u}_i a_i$. Then, due to 6 of Proposition 2.1.1 we have

$$[\![\mathbb{u} \in \mathbb{x}]\!] \geq [\![\mathbb{u}_i = \mathbb{u}]\!] \wedge [\![\mathbb{u}_i \in \mathbb{x}]\!] \geq a_i \quad \text{ for all } i.$$

We conclude that $[\![\mathbb{u} \in \mathbb{x}]\!] = I$. This proves the following:

Corollary 2.1.1. *Let $\mathbb{x}$ be a name with $[\![\mathbb{x} \neq \emptyset]\!] = I$. Suppose that $(\mathbb{u}_i)$ is a family in $\mathbb{x}\!\downarrow$ and $(a_k) \in p(I)$, then $\sum \mathbb{u}_i a_i \in \mathbb{x}\!\downarrow$.*

Remark 2.1.1. *Suppose that $\mathbb{x}$ is a name with $[\![\mathbb{x} \neq \emptyset]\!] = I$. Then we can always find an equivalent name, say $\mathbb{x}'$, with $\mathrm{dom}(\mathbb{x}') = \mathbb{x}\!\downarrow$. Indeed, take $\mathbb{x}'$ the name given by the function*

$$\mathbb{x}\!\downarrow \longrightarrow \mathcal{A} \quad \mathbb{t} \mapsto I.$$

Fix $\mathbb{t}_0 \in \mathbb{x}\!\downarrow$. For any $\mathbb{t} \in \mathrm{dom}(\mathbb{x})$ let $\mathbb{t}' := \mathbb{t}\mathbb{x}(\mathbb{t}) + \mathbb{t}_0\mathbb{x}(\mathbb{t})^c$. Then, using 6 of Proposition 2.1.1, we have

$$[\![\mathbb{t}' \in \mathbb{x}]\!] \geq [\![\mathbb{t}' = \mathbb{t}]\!] \wedge [\![\mathbb{t} \in \mathbb{x}]\!] \geq \mathbb{x}(\mathbb{t}) \wedge \mathbb{x}(\mathbb{t}) = \mathbb{x}(\mathbb{t});$$

$$[\![\mathbb{t}' \in \mathbb{x}]\!] \geq [\![\mathbb{t}' = \mathbb{t}_0]\!] \wedge [\![\mathbb{t}_0 \in \mathbb{x}]\!] \geq \mathbb{x}(\mathbb{t})^c \wedge I = \mathbb{x}(\mathbb{t})^c.$$

Consequently, $\mathbb{t}' \in \mathbb{x}\!\downarrow$, hence $[\![\mathbb{t}' \in \mathbb{x}']\!] = I$. Then, due again to 6 of Proposition 2.1.1 $[\![\mathbb{t} \in \mathbb{x}']\!] \geq [\![\mathbb{t}' \in \mathbb{x}']\!] \wedge [\![\mathbb{t} = \mathbb{t}']\!] \geq \mathbb{x}(\mathbb{t})$, and thus

$$\begin{aligned}[\![\mathbb{x} = \mathbb{x}']\!] &= \bigwedge_{\mathbb{t}\in\mathrm{dom}(\mathbb{x})} (\mathbb{x}(\mathbb{t})^c \vee [\![\mathbb{t} \in \mathbb{x}']\!]) \wedge \bigwedge_{\mathbb{t}\in\mathbb{x}\downarrow} \mathbb{x}'(\mathbb{t})^c \vee [\![\mathbb{t} \in \mathbb{x}]\!] \\ &\geq \bigwedge_{\mathbb{t}\in\mathrm{dom}(\mathbb{x})} (\mathbb{x}(\mathbb{t})^c \vee \mathbb{x}(\mathbb{t})) = I.\end{aligned}$$

Due to the remark above and [81, 2.1.9] we have the following result which will be very useful to manipulate Boolean truth values (see also [78, 2.9.(2)] and [90, Theorem 3.5]):

Theorem 2.1.4. *Let $\varphi(u)$ be a formula with one free variable u and a name $\mathbb{v}$ with $[\![\mathbb{v} \neq \emptyset]\!] = I$. Then:*

1. $[\![(\forall x \in \mathbb{v})\varphi(x)]\!] = \bigwedge_{\mathbb{u}\in\mathbb{v}\downarrow} [\![\varphi(\mathbb{u})]\!]$;
2. $[\![(\forall x \in \mathbb{v})\varphi(x)]\!] = I$ *if and only if* $[\![\varphi(\mathbb{u})]\!] = I$ *for all* $\mathbb{u} \in \mathbb{v}\!\downarrow$;
3. $[\![(\exists x \in \mathbb{v})\varphi(x)]\!] = \bigvee_{\mathbb{u}\in\mathbb{v}\downarrow} [\![\varphi(\mathbb{u})]\!]$;
4. $[\![(\exists x \in \mathbb{v})\varphi(x)]\!] = I$ *if and only if there exists* $\mathbb{u} \in \mathbb{v}\!\downarrow$ *such that* $[\![\varphi(\mathbb{u})]\!] = I$.

The language $\mathcal{L}^{(\mathcal{A})}$ enables the usual set-theoretic reasoning on the elements of $V^{(\mathcal{A})}$ and we can therefore consider within the universe $V^{(\mathcal{A})}$ all the possible mathematical objects (real numbers, functions, topological spaces, and so on). For instance, the formula "f is a function from u to v", which is abbreviated by "$f \in \mathrm{fun}(u, v)$", is a formula of the language $\mathcal{L}$. Thus, given three names $\mathbb{f}, \mathbb{u}, \mathbb{v}$ we can formulate "$\mathbb{f} \in \mathrm{fun}(\mathbb{u}, \mathbb{v})$" in the language $\mathcal{L}^{(\mathcal{A})}$. Suppose that

$[\![\mathbb{f} \in \mathrm{fun}(\mathbb{u}, \mathbb{v})]\!] = I$, then in this case we will say that $\mathbb{f}$ is a *name for a function from* $\mathbb{u}$ *to* $\mathbb{v}$. Clearly, this can be done for any definition in set theory. Therefore, we will talk about *names for vector spaces*, *names for sequences*, *names for filters* etc. without further explanations.

The language $\mathcal{L}^{(\mathcal{A})}$ and the meta-language that we use in this thesis to manipulate the universe $V^{(\mathcal{A})}$ are both based on the language of set theory. For instance, given a name $\mathbb{u}$, we can consider the singleton $\{\mathbb{u}\}$ of $\mathbb{u}$ in our mathematical discussion, but also we could consider the singleton of $\mathbb{u}$ within $V^{(\mathcal{A})}$. To avoid the possible confusion, we will denote by $\{\mathbb{u}\}_{\mathcal{A}}$ (instead of $\{\mathbb{u}\}$) the unique name in $\overline{V}^{(\mathcal{A})}$ such that $[\![\{\mathbb{u}\}_{\mathcal{A}} = \{\mathbb{u}\}]\!] = I$, which exists due to maximum principle.

In general, suppose that $\varphi(u, v_1, \dots, v_n)$ is a formula with free variables $u, v_1, \dots, v_n$ $(n \geq 0)$ in the language $\mathcal{L}$. Then, the sentence

$$(\forall y_1) \dots (\forall y_n)(\exists! x)\varphi(x, y_1, \dots, y_n)$$

is called a *uniqueness existence sentence*. Suppose that $F(v_1, \dots, v_n)$ is a n-ary function symbol $(n \geq 0)$ which is used in our mathematical discussion so that $u = F(v_1, \dots, v_n)$ abbreviates $\varphi(u, v_1, \dots, v_n)$. Let $\mathbb{v}_1, \dots, \mathbb{v}_n$ be a finite sequence of names. If the uniqueness existence sentence is true in ZFC (that is, it is a theorem), by applying first the transfer principle and then the maximum principle, we derive the existence of a unique name $\mathbb{u}$ in $\overline{V}^{(\mathcal{A})}$ such that $[\![\mathbb{u} = F(\mathbb{v}_1, \dots, \mathbb{v}_n)]\!] = I$. This representative $\mathbb{u}$ will be denoted by $F(\mathbb{v}_1, \dots, \mathbb{v}_n)_{\mathcal{A}}$ and referred to as the *interpretation* of $F(\mathbb{v}_1, \dots, \mathbb{v}_n)$ in $V^{(\mathcal{A})}$.

Let us see some examples:

- The formula $(\forall x)(x \in u \Leftrightarrow (x = v_1 \vee x = v_2))$ can be abbreviated to "$u = \{v_1, v_2\}$" and is read as "u is the doubleton of v_1 and v_2". In this case the function symbol used is $F(v_1, v_2) := \{v_1, v_2\}$. For two given names $\mathbb{v}_1, \mathbb{v}_2$, we denote by $\{\mathbb{v}_1, \mathbb{v}_2\}_{\mathcal{A}}$ the unique name in $\overline{V}^{(\mathcal{A})}$ such that $[\![\{\mathbb{v}_1, \mathbb{v}_2\}_{\mathcal{A}} = \{\mathbb{v}_1, \mathbb{v}_2\}]\!] = I$.
- The formula $(\forall x)(x \in u \Leftrightarrow (x = \{v_1\} \vee x = \{v_1, v_2\}))$ is abbreviated to "$u = (v_1, v_2)$" and is read as "u is the ordered pair of v_1 and v_2". We use the function symbol $F(v_1, v_2) := (u_1, u_2)$. Again, given two names $\mathbb{v}_1, \mathbb{v}_2$, we denote by $(\mathbb{v}_1, \mathbb{v}_2)_{\mathcal{A}}$ the unique name in $\overline{V}^{(\mathcal{A})}$ satisfying $[\![(\mathbb{v}_1, \mathbb{v}_2)_{\mathcal{A}} = (\mathbb{v}_1, \mathbb{v}_2)]\!] = I$.

Clearly, this can be done for more and more uniqueness existence sentences and function symbols. For instance, $(\mathbb{v}_1, \mathbb{v}_2, \mathbb{v}_3)_{\mathcal{A}}$ is the interpretation of the ordered triple of $\mathbb{v}_1$, $\mathbb{v}_2$ and $\mathbb{v}_3$; $(\mathbb{v}_1 \times \mathbb{v}_2)_{\mathcal{A}}$ is the interpretation of the cartesian product of $\mathbb{v}_1$ and $\mathbb{v}_2$; and so on.

Another important example is the set of natural numbers. The concepts of natural number and set of natural numbers are definable in ZFC (see Section A.4). We will denote by $\mathbb{N}_{\mathcal{A}}$ the interpretation of the set of natural numbers. The rational numbers are also definable in ZFC; we denote by $\mathbb{Q}_{\mathcal{A}}$ its interpretation. Moreover, it is known that $[\![\mathbb{N}_{\mathcal{A}} = \check{\mathbb{N}}]\!] = I$ and $[\![\mathbb{Q}_{\mathcal{A}} = \check{\mathbb{Q}}]\!] = I$ (see eg [108, pp. 129-130]).

Likewise, the real numbers are definable in ZFC. We will denote by $\mathbb{R}_{\mathcal{A}}$ the interpretation of the set of real numbers. Though, $[\![\mathbb{N}_{\mathcal{A}} = \check{\mathbb{N}}]\!] = I$ and $[\![\mathbb{Q}_{\mathcal{A}} = \check{\mathbb{Q}}]\!] = I$, in general $[\![\mathbb{R}_{\mathcal{A}} = \check{\mathbb{R}}]\!] = I$ does not hold (for a counter-example see eg [107, p. 16]).

Proposition 2.1.3. *[81, 3.2.4] Let $\mathbb{x}$, $\mathbb{y}$ be names with $[\![(\mathbb{x} \neq \emptyset) \wedge (\mathbb{y} \neq \emptyset)]\!] = I$. Then the mapping*

$$\mathbb{x}{\downarrow} \times \mathbb{y}{\downarrow} \longrightarrow (\mathbb{x} \times \mathbb{y})_{\mathcal{A}}{\downarrow} \quad (\mathbb{u}, \mathbb{v}) \mapsto (\mathbb{u}, \mathbb{v})_{\mathcal{A}}$$

is a bijection. In particular, $(\mathbb{x} \times \mathbb{y})_{\mathcal{A}}{\downarrow} = \{(\mathbb{u}, \mathbb{v})_{\mathcal{A}} \colon \mathbb{u} \in \mathbb{x}{\downarrow},\ \mathbb{v} \in \mathbb{y}{\downarrow}\}$.

Remark 2.1.2. *Let $\mathbb{x}$, $\mathbb{y}$ be names with $[\![(\mathbb{x} \neq \emptyset) \wedge (\mathbb{y} \neq \emptyset)]\!] = I$. Suppose that $(\mathbb{u}_i) \subset \mathbb{x}\!\downarrow$, $(\mathbb{v}_i) \subset \mathbb{y}\!\downarrow$, and $(a_i) \in p(I)$. We claim that*

$$\sum(\mathbb{u}_i, \mathbb{v}_i)_{\mathcal{A}} a_i = (\sum \mathbb{u}_i a_i, \sum \mathbb{v}_i a_i)_{\mathcal{A}}. \tag{2.1}$$

Indeed,

$$\begin{aligned} (2.1) \quad &\text{if and only if} \quad [\![(\mathbb{u}_j, \mathbb{v}_j) = (\sum \mathbb{u}_i a_i, \sum \mathbb{v}_i a_i)]\!] \geq a_j \text{ for all } j \\ &\text{if and only if} \quad [\![\mathbb{u}_j = \sum \mathbb{u}_i a_i]\!] \wedge [\![\mathbb{v}_j = \sum \mathbb{v}_i a_i)]\!] \geq a_j \text{ for all } j. \end{aligned}$$

The last condition is clear, hence (2.1) follows.

Definition 2.1.1. *Suppose that $\mathbb{x}, \mathbb{y}$ are names with $[\![(\mathbb{x} \neq \emptyset) \wedge (\mathbb{x} \neq \emptyset)]\!] = I$. A function $f : \mathbb{x}\!\downarrow \to \mathbb{y}\!\downarrow$ is called* extensional *if*

$$[\![\mathbb{u} = \mathbb{v}]\!] \leq [\![f(\mathbb{u}) = f(\mathbb{v})]\!] \quad \text{for all } \mathbb{u}, \mathbb{v} \in \mathbb{x}\!\downarrow.$$

Proposition 2.1.4. *[81, Theorem 3.2.12] Let $\mathbb{f}, \mathbb{x}, \mathbb{y}$ be names with $[\![(\mathbb{x} \neq \emptyset) \wedge (\mathbb{y} \neq \emptyset)]\!] = I$ and such that $[\![\mathbb{f} \in \mathrm{fun}(\mathbb{x}, \mathbb{y})]\!] = I$. Then there exists a unique function $\mathbb{f}\!\downarrow : \mathbb{x}\!\downarrow \to \mathbb{y}\!\downarrow$ such that*

$$[\![\mathbb{f}\!\downarrow(\mathbb{u}) = \mathbb{f}(\mathbb{u})]\!] = I \quad \text{for all } \mathbb{u} \in \mathbb{x}\!\downarrow.$$

Moreover, $\mathbb{f}\!\downarrow$ is extensional, $\mathbb{f}\!\downarrow$ is injective is and only in $[\![\mathbb{f}\!\downarrow \text{ is injective}]\!] = I$, $\mathbb{f}\!\downarrow$ is surjective is and only in $[\![\mathbb{f}\!\downarrow \text{ is surjective}]\!] = I$, and $\mathbb{f}\!\downarrow$ is bijective is and only in $[\![\mathbb{f}\!\downarrow \text{ is bijective}]\!] = I$.

The operation $\cdot\!\downarrow$ above is called the *descent of the name for a function.* Recall that we introduced previously a descent operation $\cdot\!\downarrow$ for a general name and we use the same notation for both. This is done by convenience and simplicity of notation. To avoid confusion, by convention, if $\mathbb{f}$ is a name for a function, the record $\mathbb{f}\!\downarrow$ will always denote the descent of a name for a function as in Proposition above.

The following results are particular cases of [81, Theorem 3.3.11]. We state them in the form given in [90].

Theorem 2.1.5. *[90, Theorem 3.6] Let $\mathbb{x}, \mathbb{y}$ be names with $[\![(\mathbb{x} \neq \emptyset) \wedge (\mathbb{y} \neq \emptyset)]\!] = I$, let $f : \mathbb{x}\!\downarrow \to \mathbb{y}\!\downarrow$ be extensional. Then there exists a name $f\!\uparrow$ for a function from $\mathbb{x}$ to $\mathbb{y}$ such that*

$$[\![f\!\uparrow(\mathbb{u}) = f(\mathbb{u})]\!] = I \quad \text{for all } \mathbb{u} \in \mathbb{x}\!\downarrow.$$

Moreover, $[\![f\!\uparrow \text{ is injective}]\!] = I$ if and only if f is injective, $[\![f\!\uparrow \text{ is surjective}]\!] = I$ if and only if f is surjective, and $[\![f\!\uparrow \text{ is bijective}]\!] = I$ if and only if f is bijective.

Let us see a variation of the result above.

Definition 2.1.2. *Let $\mathbb{x}_1, \mathbb{x}_2, \mathbb{y}$ be names with $[\![(\mathbb{x}_1 \neq \emptyset) \wedge (\mathbb{x}_2 \neq \emptyset) \wedge (\mathbb{y} \neq \emptyset)]\!] = I$. A function $f : \mathbb{x}_1\!\downarrow \times \mathbb{x}_2\!\downarrow \to \mathbb{y}\!\downarrow$ is said to be* extensional *if*

$$[\![(\mathbb{u}_1 = \mathbb{v}_1) \wedge (\mathbb{u}_2 = \mathbb{v}_2)]\!] \leq [\![f(\mathbb{u}_1, \mathbb{u}_2) = f(\mathbb{v}_1, \mathbb{v}_2)]\!] \quad \text{for all } \mathbb{u}_1, \mathbb{v}_1 \in \mathbb{x}_1\!\downarrow,\ \mathbb{u}_2, \mathbb{v}_2 \in \mathbb{x}_2\!\downarrow.$$

Theorem 2.1.6. *[90, Theorem 3.7] Let $\mathbb{x}_1, \mathbb{x}_2, \mathbb{y}$ be names with $[\![(\mathbb{x}_1 \neq \emptyset) \wedge (\mathbb{x}_2 \neq \emptyset) \wedge (\mathbb{y} \neq \emptyset)]\!] = 1$, let $f : \mathbb{x}_1\!\downarrow \times \mathbb{x}_2\!\downarrow \to \mathbb{y}\!\downarrow$ be extensional. Then there exists a name $f\!\uparrow$ for a function from $(\mathbb{x}_{\mathbb{1}} \times \mathbb{x}_2)_{\mathcal{A}}$ to $\mathbb{y}$ such that*

$$[\![f\!\uparrow(\mathbb{u}_1, \mathbb{u}_2) = f(\mathbb{u}_1, \mathbb{u}_2)]\!] = I \quad \text{for all } \mathbb{u}_1 \in \mathbb{x}_1\!\downarrow,\ \mathbb{u}_2 \in \mathbb{x}_2\!\downarrow.$$

Moreover, $[\![f\!\uparrow \text{ is injective}]\!] = I$ if and only if f is injective, $[\![f\!\uparrow \text{ is surjective}]\!] = I$ if and only if f is surjective, and $[\![f\!\uparrow \text{ is bijective}]\!] = I$ if and only if f is bijective.

Consider now a probability space $(\Omega, \mathcal{F}, P)$ and let $(\bar{\mathcal{F}}, \bar{P})$ be the corresponding probability algebra. We have that $\bar{\mathcal{F}}$ is a complete Boolean algebra. Then we can take $\mathcal{A} := \bar{\mathcal{F}}$ and consider the corresponding Boolean-valued model $V^{(\mathcal{A})}$. It is well-known that L^0 can be viewed as the interpretation of the real numbers of the universe $V^{(\mathcal{A})}$, see eg [107, Chapter 2, Section 3]. Precisely speaking, there is a canonical bijection

$$\begin{array}{rccc} i_{L^0}: & L^0 & \longrightarrow & \mathbb{R}_{\mathcal{A}}{\downarrow} \\ & \eta & \longmapsto & \eta^{\bullet} \\ & \mathbb{r}^{\circ} & \longleftarrow\!\shortmid & \mathbb{r} \end{array}$$

such that:

(i) $i_{L^0}(L^0(\mathbb{N})) = \mathbb{N}_{\mathcal{A}}{\downarrow}$;

(ii) $[\![0^{\bullet} = 0]\!] = I$, $[\![1^{\bullet} = 1]\!] = I$, $[\![(\eta+\xi)^{\bullet} = \eta^{\bullet} + \xi^{\bullet}]\!] = I$ and $[\![(\eta\xi)^{\bullet} = \eta^{\bullet}\xi^{\bullet}]\!] = I$ for all $\eta, \xi \in L^0$;

(iii) $[\![\eta^{\bullet} = \xi^{\bullet}]\!] = \{\eta = \xi\}$, $[\![\eta^{\bullet} \leq \xi^{\bullet}]\!] = \{\eta \leq \xi\}$ and $[\![\eta^{\bullet} < \xi^{\bullet}]\!] = \{\eta < \xi\}$ for all $\eta, \xi \in L^0$.

The following results exhibits the connection between sequences in L^0 and names for sequences of real numbers.

Proposition 2.1.5. *[107, Proposition 2.2.1] Suppose that $\mathbb{u}$ is a name for a sequence of real numbers, i.e. $[\![\mathbb{u} \in \mathrm{fun}(\mathbb{N}, \mathbb{R})]\!] = I$. Then*

$$\lim_n \mathbb{u}(n^{\bullet})^{\circ} = \eta \textit{ a.s. in } L^0 \quad \textit{if and only if} \quad [\![\lim_n \mathbb{u}(n) = \eta^{\circ} \text{ in } \mathbb{R}]\!] = I.$$

Suppose that (η_n) is a sequence in L^0. We can define a net $(\eta_{\mathfrak{n}})_{\mathfrak{n} \in L^0(\mathbb{N})}$ where $\eta_{\mathfrak{n}} := \sum_{k \in \mathbb{N}} 1_{\{\mathfrak{n}=k\}} \eta_k$. Then, the map $\mathbb{n} \mapsto x^{\bullet}_{\mathbb{n}^{\circ}}$ is an extensional function from $\mathbb{N}_{\mathcal{A}}{\downarrow}$ to $\mathbb{R}_{\mathcal{A}}{\downarrow}$. Theorem 2.1.5 provides us with a name $\mathbb{u}$ for a sequence of real numbers such that

$$[\![\mathbb{u}(\mathbb{n}) = \eta^{\bullet}_{\mathbb{n}^{\circ}}]\!] = I \quad \text{ for all } \mathbb{n} \in \mathbb{N}_{\mathcal{A}}{\downarrow}.$$

Then, as a direct consequence of Proposition 2.1.5 we have the following:

Proposition 2.1.6. *Suppose that (η_n) is a sequence in L^0. Then,*

$$\lim_n \eta_n = \eta \text{ a.s. in } L^0 \quad \textit{if and only if} \quad [\![\lim_n \eta^{\bullet}_{n^{\circ}} = \eta^{\bullet} \textit{ in } \mathbb{R}]\!] = I.$$

Comments 2.1.1. *If $\mathcal{A}$ is an arbitrary complete Boolean algebra, The so-called Gordon theorem (see [47]) asserts that $\mathbb{R}_{\mathcal{A}}{\downarrow}$ is a* universally complete vector lattice *(i.e. a vector lattice where every family of pairwise disjoint elements is bounded) such that $\mathcal{A}$ is isomorphic to the complete Boolean algebra of band projections in $\mathbb{R}_{\mathcal{A}}{\downarrow}$ (see [81] for terminology). In particular, in the case that $\mathcal{A} = \bar{\mathcal{F}}$ for some probability space $(\Omega, \mathcal{F}, P)$, one finds that $\mathbb{R}_{\mathcal{A}}$ is isomorphic to the universally complete vector lattice L^0. Moreover, Takeuti proved that, if $\mathcal{B}$ is the Boolean algebra of orthonormal projections in a Hilbert space, then $\mathbb{R}_{\mathcal{B}}{\downarrow}$ is isomorphic to the universally complete vector lattice of self-adjoint operators whose spectral resolutions takes values in $\mathcal{B}$.*

2.2 A precise connection between Boolean-valued convex analysis and L^0-convex analysis

Once we have introduced the main elements of Boolean-valued models, we will see how we can connect the theory of locally L^0-convex modules to this setting.

Throughout, we will consider a fixed underlying probability space $(\Omega, \mathcal{F}, P)$ and we take $\mathcal{A} := \bar{\mathcal{F}}$. Also, from now on, by an L^0-module we will mean an L^0-module with the countable concatenation property.

Next, we will define the notion of *ascent* of an L^0-module.

Suppose that E is an L^0-module:

- Given $x, y \in E$, we define $a_{x,y} := \bigvee \{a \in \mathcal{A} \colon 1_a x = 1_a y\}$.
- For any $x \in E$, let $x^\bullet$ denote the unique name in $\overline{V}^{(\mathcal{A})}$ equivalent to the name given by the function
$$\{\check{y} \colon y \in E\} \longrightarrow \mathcal{A} \quad \check{y} \mapsto a_{x,y}.$$
- We define the *ascent* of $E{\uparrow}$ to be the unique name in $\overline{V}^{(\mathcal{A})}$ equivalent to the name given by the function
$$\{x^\bullet \colon x \in E\} \longrightarrow \mathcal{A} \quad x^\bullet \mapsto I.$$

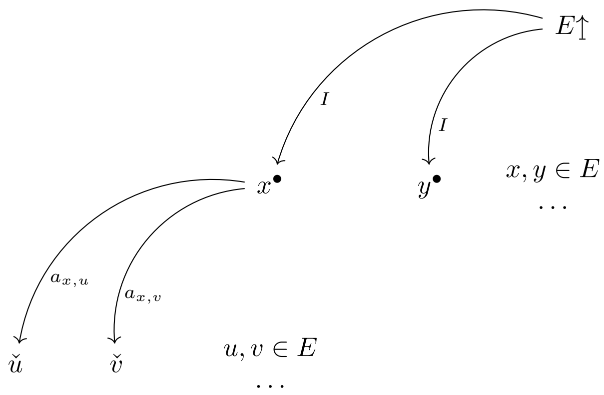

Figure 2.2: Ascent of an L^0-module.

Lemma 2.2.1. *Let E be an L^0-module. Then, given $x, y \in E$, if $a \leq a_{x,y}$, then $1_a x = 1_a y$. Moreover, for every $x, y, z \in E$, the following is fulfilled:*

1. *$a_{x,y} = a_{y,x}$;*
2. *$a_{x,y} = I$ if and only if $x = y$;*
3. *$a_{x,z} \geq a_{x,y} \wedge a_{y,z}$.*

Proof. We can find $(a_k) \in p(a_{x,y})$ with $1_{a_k} x = 1_{a_k} y$ for all $k \in \mathbb{N}$. Suppose $a \leq a_{x,y}$. For each k, take $b_k := a_k \wedge a$. Then $(b_k) \in p(a)$ and $1_{b_k} x = 1_{b_k} y$, each k. Thus $1_a x = \sum 1_{b_k} x = \sum 1_{b_k} y = 1_a y$.

Clearly, $a_{x,y} = a_{y,x}$.

If $x = y$, then obviously $a_{x,y} = I$. If $a_{x,y} = I$, then, $x = 1_{a_{x,y}} x = 1_{a_{x,y}} y = y$.

Finally, set $a := a_{x,y} \wedge a_{y,z}$. Then $1_a x = 1_a y = 1_a z$, hence $a \leq a_{x,z}$. $\square$

Remark 2.2.1. *In general, an $\mathcal{A}$-valued Boolean metric on a non-empty set X is a function $d : X \times X \to \mathcal{A}$ such that for every $x, y, z \in X$, it is satisfied (BM1) $d(x, y) = d(y, x)$; (BM2) $d(x, y) = 0$ if and only $x = y$; (BM3) $d(x, z) \leq d(x, y) \vee d(y, z)$.*

Notice that, in view of Lemma 2.2.1, the mapping $(x, y) \mapsto a^c_{x,y}$ is a Boolean metric on E.

Boolean metric spaces are closely related to Boolean-valued models (see [81, Section 3.4]). These connections will be reviewed in Chapter 3.

The following lemma will be crucial to develop this chapter.

Lemma 2.2.2. *If $x, y \in E$, then $[\![x^\bullet = y^\bullet]\!] = a_{x,y}$.*

Proof. Given $x, y \in E$, one has

$$[\![x^\bullet = y^\bullet]\!] = \bigwedge_{u\in E} (a_{x,u} \Rightarrow [\![\check{u} \in y^\bullet]\!]) \wedge \bigwedge_{v\in E} (a_{y,v} \Rightarrow [\![\check{v} \in x^\bullet]\!]) . \tag{2.2}$$

In addition,

$$[\![\check{u} \in y^\bullet]\!] = \bigvee_{w\in E} (a_{y,w} \wedge [\![\check{u} = \check{w}]\!]) = a_{y,u},$$

since $[\![\check{u} = \check{w}]\!] = I$ if $u = w$, and $[\![\check{u} = \check{w}]\!] = 0$ otherwise. Analogously, we obtain $[\![\check{v} \in x^\bullet]\!] = a_{x,v}$.

Therefore, replacing in (2.2), one has

$$[\![x^\bullet = y^\bullet]\!] = \bigwedge_{u\in E} \left(a^c_{x,u} \vee a_{y,u}\right) \wedge \bigwedge_{v\in E} \left(a^c_{y,v} \vee a_{x,v}\right) .$$

By considering above $u = x$ and $v = y$, it follows $[\![x^\bullet = y^\bullet]\!] \leq a_{x,y}$. Now, if we show that $a_{x,y} \leq a^c_{x,u} \vee a_{y,u}$ for each $u \in E$, we obtain the assertion ($a_{x,y} \leq a^c_{y,v} \vee a_{x,v}$ for each v follows by symmetry). Aiming at a contradiction, suppose that $0 < a := a_{x,y} \wedge (a^c_{x,u} \vee a_{y,u})^c$. Since $a \leq a_{x,y} \wedge a_{x,u}$ one has $1_a y = 1_a x = 1_a u$ due to Lemma 2.2.1. But $a \leq a^c_{y,u}$, hence $1_a y \neq 1_a u$, which is a contradiction. □

Proposition 2.2.1. *Let E be an L^0-module. Then the mapping $i_E : E \to E{\uparrow}{\downarrow}$, $x \mapsto x^\bullet$, is a bijection. Moreover, for any partition $(a_k) \in p(I)$ and sequence $(x_k) \subset E$ it holds*

$$i_E\left(\sum 1_{a_k} x_k\right) = \sum i_E(x_k) a_k.$$

Proof. Indeed, if $x^\bullet = y^\bullet$, then, by Lemma 2.2.2, we have $I = [\![x^\bullet = y^\bullet]\!] = a_{x,y}$; since $a_{x,y}$ is attained, it follows that $x = y$. Now, suppose that $\mathbb{z} \in E{\uparrow}{\downarrow}$. Then

$$I = [\![\mathbb{z} \in E{\uparrow}]\!] = \bigvee_{x\in E} [\![x^\bullet = \mathbb{z}]\!].$$

We can find a partition $(a_k) \in p(I)$ such that $a_k \leq [\![x^\bullet_k = \mathbb{z}]\!]$ for some $x_k \in E$, each k. The countable concatenation property yields an x so that $1_{a_k} x = 1_{a_k} x_k$ for all $k \in \mathbb{N}$. One has $[\![x^\bullet = x^\bullet_k]\!] = a_{x,x_k} \geq a_k$, each k. Due to the mixing principle, we obtain $[\![\mathbb{z} = x^\bullet]\!] = I$, and taking representatives in $\overline{V}^{(\mathcal{A})}$, we conclude that $\mathbb{z} = x^\bullet$. □

Proposition 2.2.2. *If E is an L^0-module, then $E{\uparrow}$ is a name for a vector space.*

Proof. The function

$$E{\uparrow}{\downarrow} \times E{\uparrow}{\downarrow} \longrightarrow E{\uparrow}{\downarrow}, \quad (x^\bullet, y^\bullet) \mapsto (x + y)^\bullet,$$

is well-defined due to Proposition 2.2.1. Besides, for given $x_1, x_2, y_1, y_2 \in E$, it holds that

$$[\![(x^\bullet_1, y^\bullet_1) = (x^\bullet_2, y^\bullet_2)]\!] = [\![(x^\bullet_1 = x^\bullet_2) \wedge (y^\bullet_1 = y^\bullet_2)]\!] = [\![x^\bullet_1 = x^\bullet_2]\!] \wedge [\![y^\bullet_1 = y^\bullet_2]\!]$$

$$= a_{x_1,x_2} \wedge a_{y_1,y_2} \leq a_{x_1+y_1,x_2+y_2} = [\![(x_1+y_1)^\bullet = (x_2+y_2)^\bullet]\!].$$

Due to Theorem 2.1.6, we can define a name $+ := +\!\uparrow$ with

$$[\![+ \in \mathrm{fun}(E\!\uparrow \times E\!\uparrow, E\!\uparrow)]\!] = I$$

and such that

$$[\![(x+y)^\bullet = x^\bullet + y^\bullet]\!] = I \quad \text{for all } x, y \in E.$$

Similarly, we consider the function

$$\mathbb{R}_{\mathcal{A}}\!\downarrow \times E\!\uparrow\downarrow \longrightarrow E\!\uparrow\downarrow, \quad (\mathbb{r}, x^\bullet) \mapsto (\mathbb{r}^\circ x)^\bullet.$$

If $\mathbb{r}_1, \mathbb{r}_2 \in \mathbb{R}_{\mathcal{A}}\!\downarrow$ and $x_1, x_2 \in E$, then

$$[\![(\mathbb{r}_1, x_1^\bullet) = (\mathbb{r}_2, x_2^\bullet)]\!] = [\![(\mathbb{r}_1 = \mathbb{r}_2) \wedge (x_1^\bullet = x_2^\bullet)]\!] = [\![\mathbb{r}_1 = \mathbb{r}_2]\!] \wedge [\![x_1^\bullet = x_2^\bullet]\!]$$

$$= \{\mathbb{r}_1^\circ = \mathbb{r}_2^\circ\} \wedge a_{x_1,x_2} \leq a_{\mathbb{r}_1^\circ x_1, \mathbb{r}_2^\circ x_2} = [\![(\mathbb{r}_1^\circ x_1)^\bullet = (\mathbb{r}_2^\circ x_2)^\bullet]\!].$$

Again, due to Theorem 2.1.6, we can define a name $\cdot := \cdot\!\uparrow$ with

$$[\![\cdot \in \mathrm{fun}(\mathbb{R} \times E\!\uparrow, E\!\uparrow)]\!] = I$$

and such that

$$[\![(\eta x)^\bullet = \eta^\bullet x^\bullet]\!] = I \quad \text{for all } x \in E,\ \eta \in L^0.$$

These relations show that $+$, $\cdot$ are names for operations on $E\!\uparrow$. Further, bearing in mind Theorem 2.1.4, it can be checked that $(E\!\uparrow, +, \cdot)_{\mathcal{A}}$ is a name for a vector space. $\square$

Proposition 2.2.3. *If $\mathbb{X}$ is a name for a vector space, then $\mathbb{X}\!\downarrow$ is an L^0-module. Moreover, if $(a_k) \in p(I)$ and $(\mathbb{x}_k) \subset \mathbb{X}\!\downarrow$, then $\sum 1_{a_k}\mathbb{x}_k = \sum \mathbb{x}_k a_k$.*

Proof. For $\mathbb{x}, \mathbb{y} \in \mathbb{X}\!\downarrow$ and $\eta \in L^0$, the maximum principle provides us with $\mathbb{u} \in \mathbb{X}\!\downarrow$ such that $[\![\mathbb{x} + \mathbb{y} = \mathbb{u}]\!] = I$, we define $\mathbb{x} + \mathbb{y} := \mathbb{u}$. Similarly, we put $\eta\mathbb{x} := \mathbb{v}$ the unique name $\mathbb{v} \in \mathbb{X}\!\downarrow$ so that $[\![\eta^\bullet \mathbb{x} = \mathbb{v}]\!] = I$. It follows by inspection that $\mathbb{X}\!\downarrow$ endowed with these operations is an L^0-module possibly without the countable concatenation property. To prove that $\mathbb{X}\!\downarrow$ has the countable concatenation property, take $(a_k) \in p(I)$ and $(\mathbb{u}_k) \subset \mathbb{X}\!\downarrow$ and let $\mathbb{u} := \sum \mathbb{u}_k a_k$. Since $[\![\mathbb{u} = \mathbb{u}_k]\!] \geq a_k$ and $[\![\mathbb{u}_k \in \mathbb{X}]\!] = I$, due to Corollary 2.1.1 it follows that $\mathbb{u} \in \mathbb{X}\!\downarrow$. One has $[\![1_{a_k}^\bullet = 1]\!] = a_k$ and $[\![1_{a_k}^\bullet = 0]\!] = a_k^c$ for each $k \in \mathbb{N}$. Then

$$[\![1_{a_k}^\bullet \mathbb{u} = 1_{a_k}^\bullet \mathbb{u}_k]\!] \geq [\![((\mathbb{u} = \mathbb{u}_k) \wedge (1_{a_k}^\bullet = 1)) \vee (1_{a_k}^\bullet = 0)]\!]$$

$$= ([\![\mathbb{u} = \mathbb{u}_k]\!] \wedge [\![1_{a_k}^\bullet = 1]\!]) \vee [\![1_{a_k}^\bullet = 0]\!] = a_k \vee a_k^c = I \quad \text{for all } k \in \mathbb{N}.$$

Therefore $1_{a_k}\mathbb{u} = 1_{a_k}\mathbb{u}_k$ for each $k \in \mathbb{N}$ and the assertion follows. $\square$

Proposition 2.2.4. *Suppose that E, F are L^0-modules and let $f : E \to F$ be and L^0-module homomorphism. Then there exists a name $f\!\uparrow$ for a linear function from $E\!\uparrow$ to $F\!\uparrow$ such that*

$$[\![f\!\uparrow(x^\bullet) = f(x)^\bullet]\!] = I \quad \text{for all } x \in E.$$

Moreover, if f is a monomorphism then $[\![f\!\uparrow \text{ is a monomorphism}]\!] = I$; if f is an epimorphism then $[\![f\!\uparrow \text{ is an epimorphism}]\!] = I$; and if f is an isomorphism then $[\![f\!\uparrow \text{ is an isomorphism}]\!] = I$.

Proof. We consider $(i_F \circ f \circ i_E^{-1}) : E{\uparrow}{\downarrow} \to F{\uparrow}{\downarrow}$, $x^\bullet \mapsto f(x)^\bullet$. Using that f is a homomorphism we have that

$$[\![x^\bullet = y^\bullet]\!] = a_{x,y} \leq a_{f(x),f(y)} = [\![f(x)^\bullet = f(y)^\bullet]\!] \quad \text{for all } x, y \in E.$$

This shows that $(i_F \circ f \circ i_E^{-1})$ is extensional and, according to Theorem 2.1.5, there exists a name $f{\uparrow} := (i_F \circ f \circ i_E^{-1}){\uparrow}$ for a function from $E{\uparrow}$ to $F{\uparrow}$ such that

$$[\![f(x)^\bullet = f{\uparrow}(x^\bullet)]\!] = I \quad \text{for all } x \in E.$$

In particular, we have that

$$[\![f{\uparrow}(x^\bullet + y^\bullet) = f{\uparrow}(x^\bullet) + f{\uparrow}(y^\bullet)]\!] = I \text{ and } [\![f{\uparrow}(\mathbb{r}x^\bullet) = \mathbb{r}f{\uparrow}(x^\bullet)]\!] = I \text{ for all } x, y \in E,\ \mathbb{r} \in \mathbb{R}_{\mathcal{A}}{\downarrow}.$$

Due to Theorem 2.1.4, we can conclude that $f{\uparrow}$ is a name for a linear function from $E{\uparrow}$ to $F{\uparrow}$. In addition, due also to Theorem 2.1.4 if f is injective (resp. surjective), then $f{\uparrow}$ is a name for an injection (resp. a surjection). □

Proposition 2.2.5. *Let $\mathbb{X}, \mathbb{Y}$ be names for vector spaces and suppose that $\mathbb{f}$ is a name for a linear function from $\mathbb{X}$ to $\mathbb{Y}$. Then there exists a unique function $\mathbb{f}{\downarrow} : \mathbb{X}{\downarrow} \to \mathbb{Y}{\downarrow}$ such that $[\![\mathbb{f}{\downarrow}(\mathbb{x}) = \mathbb{f}(\mathbb{x})]\!] = I$ for all $\mathbb{x} \in \mathbb{X}{\downarrow}$. Furthermore, $\mathbb{f}{\downarrow}$ is an L^0-module homomorphism and if $[\![\mathbb{f}$ is a monomorphism$]\!] = I$ then $\mathbb{f}{\downarrow}$ is a monomorphism; if $[\![\mathbb{f}$ is an epimorphism$]\!] = I$ then $\mathbb{f}{\downarrow}$ is an epimorphism; and if $[\![\mathbb{f}$ is an isomorphism$]\!] = I$ then $\mathbb{f}{\downarrow}$ is an isomorphism;*

Proof. Given $\mathbb{x}, \mathbb{y} \in \mathbb{X}{\downarrow}$, we have that $[\![\mathbb{f}(\mathbb{x} + \mathbb{y}) = \mathbb{f}(\mathbb{x}) + \mathbb{f}(\mathbb{y})]\!] = I$. Taking representatives in $\overline{V}^{(\mathcal{A})}$, one has that

$$\mathbb{f}{\downarrow}(\mathbb{x} + \mathbb{y}) = \mathbb{f}(\mathbb{x} + \mathbb{y}) = \mathbb{f}(\mathbb{x}) + \mathbb{f}(\mathbb{y}) = \mathbb{f}{\downarrow}(\mathbb{x}) + \mathbb{f}{\downarrow}(\mathbb{y}).$$

Now, given $\eta \in L^0$ and $\mathbb{x} \in \mathbb{X}{\downarrow}$, we have that

$$[\![\mathbb{f}(\eta^\bullet \mathbb{x}) = \eta^\bullet \mathbb{f}(\mathbb{x})]\!] = I.$$

Taking representatives in $\overline{V}^{(\mathcal{A})}$, we obtain that

$$\mathbb{f}{\downarrow}(\eta \mathbb{x}) = \mathbb{f}(\eta^\bullet \mathbb{x}) = \eta^\bullet \mathbb{f}(\mathbb{x}) = \eta \mathbb{f}{\downarrow}(\mathbb{x}).$$

□

Proposition 2.2.6. *If E is an L^0-module, then the mapping $i_E : E \to E{\uparrow}{\downarrow}$, $x \mapsto x^\bullet$ is an isomorphism of L^0-modules.*

Proof. By Proposition 2.2.1 we have that $E \to E{\uparrow}{\downarrow}$, $x \mapsto x^\bullet$ is a bijection. It is clear that is an L^0-module homomorphism. Thus, it is an isomorphism of L^0-modules. □

Proposition 2.2.7. *Let $\mathbb{X}$ be a name for a vector space. Then there exists a name $\mathbb{i}_{\mathbb{X}}$ for an isomorphism of vector spaces between $\mathbb{X}$ and $\mathbb{X}{\downarrow}{\uparrow}$ with $[\![\mathbb{i}_{\mathbb{X}}(\mathbb{x}) = \mathbb{x}^\bullet]\!]$ for all $\mathbb{x} \in \mathbb{X}{\downarrow}$.*

Proof. Due to Proposition 2.2.6, the function $i_{\mathbb{X}\downarrow}$ is an isomorphism of L^0-modules. Then $\mathbb{i}_{\mathbb{X}} := i_{\mathbb{X}\downarrow}{\uparrow}$ is a name for an isomorphism of vector spaces. □

The following result summarizes the results provided above by using the language of categories:

Theorem 2.2.1. *Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{A} := \bar{\mathcal{F}}$. There is an equivalence of categories between the category of names of vector spaces and linear functions in $\overline{V}^{(\mathcal{A})}$, and the category of L^0-modules and L^0-module homomorphisms.*

Proof. If $\mathbb{X}$ is a name for a vector space and $\mathbb{f}$ is a name for a linear function between vector spaces, we define $\Phi(\mathbb{X}) := \mathbb{X}\downarrow$ and $\Phi(\mathbb{f}) := \mathbb{f}\downarrow$.

Propositions 2.2.3 and 2.2.5 tell us that Φ is a functor from the category of names for vector spaces and linear functions to the category of L^0-modules and L^0-module homomorphisms.

If E is an L^0-module and f is an L^0-module homomorphism, we put $\Psi(E) = E\uparrow$, $\Psi(f) := f\uparrow$. Propositions 2.2.2 and 2.2.4 show that Ψ is a functor from the category of L^0-modules and L^0-module homomorphisms to the category for names of vector spaces and linear functions.

Proposition 2.2.6 shows that i_E is an isomorphism of L^0-modules for any L^0-module E. Moreover, a close look shows that it defines a natural isomorphism between $\Phi\Psi$ and the identity functor.

$$
\begin{tikzcd}
E \arrow[rrr, "f"] \arrow[ddd, "i_E"'] & & & F \arrow[ddd, "i_F"] \\
& x \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dr, phantom, "\circlearrowleft"] & f(x) \arrow[d, mapsto] & \\
& x^\bullet \arrow[r, mapsto] & f(x)^\bullet & \\
E\uparrow\downarrow \arrow[rrr, "f\uparrow\downarrow"'] & & & F\uparrow\downarrow
\end{tikzcd}
$$

Similarly, $\Psi\Phi$ is naturally isomorphic to the identity functor (see Proposition 2.2.7).

$$
\begin{tikzcd}
\mathbb{X} \arrow[rrr, "\mathbb{f}"] \arrow[ddd, "\mathbb{i}_{\mathbb{X}}"'] & & & \mathbb{Y} \arrow[ddd, "\mathbb{i}_{\mathbb{Y}}"] \\
& \mathbb{x} \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{f}(\mathbb{x}) \arrow[d, mapsto] & \\
& \mathbb{x}^\bullet \arrow[r, mapsto] & \mathbb{f}(\mathbb{x})^\bullet & \\
\mathbb{X}\downarrow\uparrow \arrow[rrr, "\mathbb{f}\downarrow\uparrow"'] & & & \mathbb{Y}\downarrow\uparrow
\end{tikzcd}
$$

□

Next, we will define new types of ascent operations for stable subsets and stable collections of stable subsets of an L^0-module:

Let E be an L^0-module, we define:

- *Ascent of a stable subset*: given a stable subset S of E we define the ascent of S, denoted by $S\uparrow$, to be the unique member of $\overline{V}^{(\mathcal{A})}$ equivalent to the name

$$\{x^\bullet \colon x \in S\} \longrightarrow \mathcal{A} \quad x^\bullet \mapsto I;$$

- *Ascent of a stable collection of stable subsets*: given a stable collection $\mathscr{C}$ of stable subsets of E, we define its ascent, denoted by $\mathscr{C}\Uparrow$, to be the unique member of $\overline{V}^{(\mathcal{A})}$ equivalent to

$$\{S\uparrow \colon S \in \mathscr{C}\} \longrightarrow \mathcal{A} \quad S\uparrow \mapsto I.$$

Proposition 2.2.8. *Let E be an L^0-module and S a stable subset. Then $S\uparrow$ is a name for a non-empty subset of $E\uparrow$. Moreover, $x \mapsto x^\bullet$ is a bijection from S to $S\uparrow\downarrow$.*

Proof. By means of Theorem 2.1.4, it is straightforward to prove that $S\uparrow$ is a name for a non-empty subset of $E\uparrow$. The injectivity of $x \mapsto x^\bullet$ is known from Proposition 2.2.6. For the surjectivity, if $\mathbbm{x} \in S\uparrow\downarrow$, then

$$I = [\![\mathbbm{x} \in S\uparrow]\!] = \bigvee\{[\![\mathbbm{x} = x^\bullet]\!] \colon x \in S\}.$$

Take $(a_k) \in p(I)$ and $(x_k) \subset S$ with $a_k \leq [\![\mathbbm{x} = x_k^\bullet]\!]$, each k. Set $x := \sum 1_{a_k} x_k$. Since S is stable, $x \in S$. Due to Proposition 2.2.6, we have that $x^\bullet = \sum x_k^\bullet a_k = \mathbbm{x}$. $\square$

Lemma 2.2.3. *Let $\mathscr{C}$ be a stable collection of stable subsets of E. Then*

$$[\![U\uparrow = V\uparrow]\!] = a_{U,V} \quad \text{for all } U, V \in \mathscr{C},$$

where $a_{U,V} := \bigvee\{a \in \mathcal{A} \colon 1_a U = 1_a V\}$ for $U, V \in \mathscr{C}$.

Proof. Fix $U, V \in \mathscr{U}$. We have

$$[\![U\uparrow = V\uparrow]\!] = \bigwedge_{x\in U} [\![x^\bullet \in V\uparrow]\!] \wedge \bigwedge_{y \in V} [\![y^\bullet \in U\uparrow]\!] = \bigwedge_{x\in U}\bigvee_{y\in V} a_{x,y} \wedge \bigwedge_{y\in V}\bigvee_{x\in U} a_{x,y}. \tag{2.3}$$

For every $x \in U$, we can find a partition $(a_k) \in p(a_{U,V})$ such that $1_{a_k} U = 1_{a_k} V$, each k. Then we have a sequence $(y_k) \subset V$ such that $1_{a_k} x = 1_{a_k} y_k$ for each $k \in \mathbb{N}$. Take some $y_0 \in V$ and set $y_x := 1_{a_{U,V}^c} y_0 + \sum 1_{a_k} y_k$. Since V is stable, we have that $y_x \in V$, further $1_{a_{U,V}} x = 1_{a_{U,V}} y_x$. For each $x \in U$, we construct a similar $y_x \in V$. Then $a_{U,V} \leq a_{x,y_x}$ for all $x \in U$. Similarly, for every $y \in V$ we can find $x_y \in U$ with $a_{U,V} \leq a_{x_y,y}$. Bearing in mind (2.3), we can conclude that $a_{U,V} \leq [\![U\uparrow = V\uparrow]\!]$.

By using again that V is stable, for each $x \in U$ one can find $y^x \in V$ such that $\bigvee_{y\in V} a_{x,y} = a_{x,y^x}$. Similarly, for every $y \in V$ one can pick up $x^y \in U$ with $\bigvee_{x\in U} a_{x,y} = a_{x^y,y}$. By using this in (2.3), we get that always

$$[\![U\uparrow = V\uparrow]\!] \leq a_{x,y^x} \quad \text{and} \quad [\![U\uparrow = V\uparrow]\!] \leq a_{y,x^y},$$

and, hence we find that

$$1_{[\![U\uparrow = V\uparrow]\!]} x = 1_{[\![U\uparrow = V\uparrow]\!]} y^x \quad \text{for every } x \in U,$$

$$1_{[\![U\uparrow = V\uparrow]\!]} x^y = 1_{[\![U\uparrow = V\uparrow]\!]} y \quad \text{for every } y \in V.$$

It follows that $1_{[\![U\uparrow = V\uparrow]\!]} U = 1_{[\![U\uparrow = V\uparrow]\!]} V$, and therefore $[\![U\uparrow = V\uparrow]\!] \leq a_{U,V}$. $\square$

Proposition 2.2.9. *Let E be an L^0-module and $\mathscr{C}$ a stable collection of stable subsets of E. Then $\mathscr{C}{\Uparrow}$ is a name for a non-empty collection of non-empty subsets of $E{\uparrow}$. Moreover, the map $S \mapsto S{\uparrow}{\downarrow}$ is a bijection from $\mathscr{C}$ to $\mathscr{C}{\Uparrow}{\Downarrow}$.*

Proof. Bearing in mind Theorem 2.1.4, it is straightforward to check that $\mathscr{C}{\Uparrow}$ is a name for a non-empty collection of non-empty subsets of $E{\uparrow}$.

The image of any $S \in \mathscr{C}$ via the mapping $x \mapsto x^\bullet$ is $S{\uparrow}{\downarrow}$ due to Proposition 2.2.8. Now we claim that the assignment $S \mapsto S{\uparrow}{\downarrow}$ is a bijection from $\mathscr{C}$ to $\mathscr{C}{\Uparrow}{\Downarrow}$. The assignment is injective because $x \mapsto x^\bullet$ is injective. Let us show that it is surjective. If $\mathbb{W}{\downarrow} \in \mathscr{C}{\Uparrow}{\Downarrow}$ with $\mathbb{W} \in \mathscr{C}{\Uparrow}{\downarrow}$, then

$$I = [\![\mathbb{W} \in \mathscr{C}{\Uparrow}]\!] = \bigvee_{S \in \mathscr{C}} [\![\mathbb{W} = S{\uparrow}]\!].$$

Take a maximal partition $(b_k) \in p(I)$ and $(S_k) \subset \mathscr{C}$ such that $b_k \leq [\![S_k{\uparrow} = \mathbb{W}]\!]$ for all $k \in \mathbb{N}$. Set $S := \sum 1_{a_k} S_k \in \mathscr{C}$. Since $\mathscr{C}$ is stable, we have that $S \in \mathscr{C}$. Let us prove that S is a preimage for $\mathbb{W}{\downarrow}$. Due to Lemma 2.2.3, we have that $[\![S{\uparrow} = S_k{\uparrow}]\!] \geq b_k$ for each $k \in \mathbb{N}$. By the mixing principle, it follows that $[\![S{\uparrow} = \mathbb{W}]\!] = I$. We conclude that $S{\uparrow}{\downarrow} = \mathbb{W}{\downarrow}$, as desired. □

Proposition 2.2.10. *Let E be an L^0-module. If $\mathbb{S}$ is a name for a non-empty subset of $E{\uparrow}$, then $\mathbb{S}{\downarrow}$ is a stable subset of $E{\uparrow}{\downarrow}$.*

Proof. Suppose that $[\![\emptyset \neq \mathbb{S} \subset E{\uparrow}]\!] = I$. It is clear that $\mathbb{S}{\downarrow} \subset E{\uparrow}{\downarrow}$. Finally, $\mathbb{S}{\downarrow}$ is stable due to the mixing principle and Corollary 2.1.1. □

Proposition 2.2.11. *Let E be an L^0-module. If $\mathbb{v}$ is a name for a non-empty collection of subsets of $E{\uparrow}$, then $\mathbb{v}{\Downarrow}$ is a stable collection of stable subsets of $E{\uparrow}{\downarrow}$.*

Proof. Suppose that $[\![\emptyset \neq \mathbb{v} \subset \mathcal{P}(E{\uparrow}) \setminus \{\emptyset\}]\!] = I$. Due Proposition 2.2.10 we have that $\mathbb{v}{\Downarrow}$ is a non-empty collection of stable subsets of $E{\uparrow}{\downarrow}$. Let us prove that it is stable. Take a sequence $(\mathbb{S}_k)$ of members of $\mathbb{v}{\downarrow}$ and $(a_k) \in p(I)$. Set $\mathbb{S} := \sum \mathbb{S}_k a_k$. Then $\mathbb{S} \in \mathbb{v}{\downarrow}$ due to Corollary 2.1.1. We claim that $1_{a_k}\mathbb{S}{\downarrow} = 1_{a_k}\mathbb{S}_k{\downarrow}$ for all $k \in \mathbb{N}$. Indeed

$$\begin{aligned} 1_{a_k}\mathbb{S}{\downarrow} = 1_{a_k}\mathbb{S}_k{\downarrow} \quad & \text{if and only if} \quad [\![1^\bullet_{a_k}\mathbb{S} = 1^\bullet_{a_k}\mathbb{S}_k]\!] = I \\ & \text{if and only if} \quad [\![((\mathbb{S} = \mathbb{S}_k) \wedge (1^\bullet_{a_k} = 1)) \vee (1^\bullet_{a_k} = 0)]\!] = I \\ & \text{if and only if} \quad ([\![\mathbb{S} = \mathbb{S}_k]\!] \wedge a_k) \vee a_k^c = I. \end{aligned}$$

But the last condition holds as $[\![\mathbb{S} = \mathbb{S}_k]\!] \geq a_k$. This proves that $\mathbb{v}{\Downarrow}$ is stable. □

In what follows we will turn to the study of topological aspects.

Suppose that $(\mathbb{X}, \mathbb{T})_{\mathcal{A}}$ is a name for a locally convex space; that is, $\mathbb{X}$ is a name for a vector space and $\mathbb{T}$ is a name for a locally convex topology on $\mathbb{X}$. Any locally convex space has a neighborhood base of the origin consisting of convex neighborhoods. Then, by applying first the transfer principle and then the maximum principle, we can derive the existence of a name $\mathbb{v}$ for a neighborhood base of the origin consisting of convex subsets, that is

$$[\![(\forall U \in \mathbb{v})(U \text{ “is convex”})]\!] = I.$$

We will use the following notation convention. Let E be an L^0-module. If $\mathscr{U}$ is a neighborhood base of $0 \in E$ for a locally stable L^0-convex topology $\mathscr{T}$, we will write $\langle \mathscr{U} \rangle := \mathscr{T}$. Suppose that $(\mathbb{X}, \mathbb{T})_{\mathcal{A}}$ is a name for a locally convex space and $\mathbb{v}$ is a name for a neighborhood base of the origin, we will use the notation $\langle \mathbb{v} \rangle_{\mathcal{A}} := \mathbb{T}$.

Proposition 2.2.12. *Let $(\mathbb{X}, \mathbb{T})_{\mathcal{A}}$ be a name for a locally convex space and suppose that $\mathbb{V}$ is a name for a neighborhood base of the origin consisting of convex subsets. Then $\mathbb{V}\Downarrow$ is a neighborhood base of the origin for a locally stable L^0-convex topology $\langle \mathbb{V}\Downarrow \rangle$ on $\mathbb{X}\downarrow$.*

Proof. Due to Proposition 2.2.11, $\mathbb{V}\Downarrow$ is a stable collection of stables subsets of the L^0-module $\mathbb{X}\downarrow$. By means of Theorem 2.1.4, it can be proven that each $\mathbb{U} \in \mathbb{V}\Downarrow$ is L^0-convex and $\mathbb{V}\Downarrow$ is a neighborhood base of the origin for a topology for which the sum and the product by elements of L^0 are continuous. Therefore $(\mathbb{X}\downarrow, \langle \mathbb{V}\Downarrow \rangle)$ is a locally stable L^0-convex module. □

Proposition 2.2.13. *Let $(E, \mathscr{T})$ be a locally stable L^0-convex module. If $\mathscr{U}$ is a stable neighborhood base of $0 \in E$ consisting of stable and L^0-convex sets, then $\mathscr{U}\Uparrow$ is a name for a neighborhood base of the origin of a locally convex topology $\langle \mathscr{U}\Uparrow \rangle_{\mathcal{A}}$ on $E\uparrow$.*

Proof. Consider the name $\mathscr{U}\Uparrow$. Then, Using Theorem 2.1.4, it simple to verify that each $\mathbb{U} \in \mathscr{U}\Uparrow\downarrow$ is a name for a convex subset of $E\uparrow$, and that $\mathscr{U}\Uparrow$ is a name for a neighborhood base of the origin of some locally convex topology on $E\uparrow$. □

Proposition 2.2.14. *Let $(\mathbb{X}_1, \mathbb{T}_1)_{\mathcal{A}}$, $(\mathbb{X}_2, \mathbb{T}_2)_{\mathcal{A}}$ be names for locally convex spaces and suppose that $\mathbb{f}$ is a name for a continuous linear function from $\mathbb{X}_1$ to $\mathbb{X}_2$. If $\mathbb{V}_1$ and $\mathbb{V}_2$ are names for neighborhood bases of the origin for $\mathbb{T}_1$ and $\mathbb{T}_2$, respectively, then $\mathbb{f}\downarrow : \mathbb{X}_1\downarrow \to \mathbb{X}_2\downarrow$ is a continuous L^0-module homomorphism with respect to the topologies $\langle \mathbb{V}_1\Downarrow \rangle$ and $\langle \mathbb{V}_2\Downarrow \rangle$.*

Proof. Consider $\mathbb{f}\downarrow$. Due to Proposition 2.2.5 is an L^0-module homomorphism between $\mathbb{X}_1\downarrow$ and $\mathbb{X}_2\downarrow$ such that $[\![\mathbb{f}\downarrow(\mathbb{x}) = \mathbb{f}(\mathbb{x})]\!] = I$ for all $\mathbb{x} \in \mathbb{X}_1\downarrow$. The function $\mathbb{f}\downarrow$ is also continuous. To see this, suppose that $\mathbb{T}_1 = \langle \mathbb{V}_1 \rangle$ and $\mathbb{T}_2 = \langle \mathbb{V}_2 \rangle$ and take $\mathscr{T}_1 := \langle \mathbb{V}_1\Downarrow \rangle$ and $\mathscr{T}_2 := \langle \mathbb{V}_2\Downarrow \rangle$. Consider $\mathbb{U} \in \mathbb{V}_2\downarrow$. Then, since $\mathbb{f}$ is a name of a continuous function, there is $\mathbb{W} \in \mathbb{V}_1\downarrow$ such that $[\![\mathbb{f}(\mathbb{W}) \subset \mathbb{U}]\!] = I$. Then $\mathbb{f}\downarrow(\mathbb{W}\downarrow) \subset \mathbb{U}\downarrow$ and this proves that $\mathbb{f}\downarrow$ is continuous. □

Proposition 2.2.15. *Let $(E_1, \mathscr{T}_1)$, $(E_2, \mathscr{T}_2)$ be locally stable L^0-convex modules and let $f : E_1 \to E_2$ be a continuous L^0-module homomorphism. If $\mathscr{U}_1$ and $\mathscr{U}_2$ are stable neighborhood of $0 \in E_1$ and $0 \in E_2$, respectively, consisting of stable subsets, then $f\uparrow$ is a name for a continuous linear function from $E_1\uparrow$ to $E_2\uparrow$ with respect to $\langle \mathscr{U}_1\Uparrow \rangle_{\mathcal{A}}$ and $\langle \mathscr{U}_2\Uparrow \rangle_{\mathcal{A}}$.*

Proof. Due to Proposition 2.2.4, we know that $f\uparrow : E_1\uparrow \to E_2\uparrow$ is the name for a linear function. It is in fact the name for a continuous linear functional, because if two basic neighborhoods U, V satisfy $f(U) \subset V$, then $[\![f\uparrow(U\uparrow) \subset V\uparrow]\!] = I$. □

Proposition 2.2.16. *Let $(E, \mathscr{T})$ be a locally stable L^0-module and $\mathscr{U}$ a stable neighborhood of $0 \in E$ consisting of stable subsets. Then $i_E : E \to E\uparrow\downarrow$, $x \mapsto x^\bullet$, is an isomorphism of locally stable L^0-convex modules between $(E, \mathscr{T})$ and $(E\uparrow\downarrow, \langle \mathscr{U}\Uparrow\Downarrow \rangle)$; that is, it is an isomorphism of L^0-modules which is a homeomorphism.*

Proof. Proposition 2.2.6 shows that i_E is an isomorphism of L^0-modules. Proposition 2.2.9 gives a bijection between neighborhood basis of the origin. The assertion follows. □

Proposition 2.2.17. *Let $(\mathbb{X}, \mathbb{T})_{\mathcal{A}}$ be a name for a locally convex space, and $\mathbb{V}$ a name for a neighborhood of the origin. Then $\mathbb{i}_{\mathbb{X}}$ is name for a isomorphism of locally convex spaces between $(\mathbb{X}, \mathbb{T})_{\mathcal{A}}$ and $(\mathbb{X}\downarrow\uparrow, \langle \mathbb{V}\Downarrow\Uparrow \rangle)_{\mathcal{A}}$.*

Proof. We know that $\mathbb{i}_{\mathbb{X}}$ is a name for an isomorphism of vector spaces due to Proposition 2.2.16. Clearly, it is also a homeomorphism. □

The following result provides a connection between locally stable L^0-convex modules and names of locally convex spaces:

Theorem 2.2.2. *Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{A} = \bar{\mathcal{F}}$. There is an equivalence of categories between the category of names for locally convex spaces and continuous linear functions in $\overline{V}^{(\mathcal{A})}$, and the category of locally stable L^0-convex modules and continuous L^0-module homomorphisms.*

Proof. We consider the same functor Φ as in Theorem 2.2.1, but restricted to the category of names for locally convex spaces and continuous linear functions in $\overline{V}^{(\mathcal{A})}$. Then Propositions 2.2.12 and 2.2.14 yield that Φ is a functor from the first category to the second one.

Propositions 2.2.13 and 2.2.15 show that Ψ is a functor from the second category to the first one.

Given a locally L^0-convex module $(E, \mathscr{T})$, Proposition 2.2.17 tells us that the function i_E is an isomorphism of locally stable L^0-convex modules for any locally stable L^0-convex module. Given a name $(\mathbb{X}, \mathbb{T})_{\mathcal{A}}$ for a locally convex space, then $\mathbb{i}_{\mathbb{X}}$ is a name for an isomorphism of locally convex spaces due to Proposition 2.2.17. They are natural transformation, hence the assertion follows. $\square$

Comments 2.2.1. *Theorem 2.2.1 is not new in literature. Indeed, Gordon [48] provided a more general equivalence between the category of names for vector spaces and names for linear functions in $V^{(\mathcal{A})}$ and the category of unital separated injective K-modules and K-module morphisms, where K is a rationally complete semiprime commutative ring and $\mathcal{A}$ is the complete Boolean-algebra of annihilator ideals (see [81] for terminology). Besides, it can be checked that the countable concatenation property of an L^0-module E is equivalent to the injectivity of E. Thus, we find that Theorem 2.2.1 is covered by the mentioned equivalence of categories. However, to our knowledge, Theorem 2.2.2 seems to be new in literature.*

A Kantorovich space *is a Dedekind complete archimedean vector lattice. Due to the Gordon theorem on Boolean-valued real numbers (see [81, 5.2.2]), any universally complete Kantorovich space K with order unity is the descent of the real numbers of some Boolean-valued model $V^{(\mathcal{A})}$. Beside, due to the maximum principle, K inherits the ring operations of $\mathbb{R}^{(\mathcal{A})}$ and becomes an f-algebra. Thus, given a universally complete Kantorovich space K with order unity, we can define the notion of* locally stable K-module *(changing countable partitions by arbitrary partitions in Definition 1.4.3) and, by adapting the proof of Theorem 2.2.2, we can provide a similar equivalence between the category of locally stable K-modules and continuous K-module homomorphisms and the category of names of locally convex spaces and linear continuous functions in $\overline{V}^{(\mathcal{A})}$. However, we do not provide the proof as this work is focused on applications to mathematical finance which are covered by the order separable universally complete Kantorovich space L^0.*

2.3 A Boolean-valued transfer principle from classical convex analysis to L^0-convex analysis

The transfer principle of Boolean-valued models proves that, for any theorem φ, the assertion "$\|\varphi\| = I$" also holds. This is a machine to produce a theorem from another theorem. If, given a theorem φ of convex analysis, we manage to interpret $\|\varphi\| = I$ as a statement of L^0-convex analysis, we will have proven in fact a new theorem of L^0-convex analysis. In this section, we will see what kind of theorems we can obtain by this method in view of the connections studied in the previous section.

Let us start with an observation. Recall the canonical bijection $\eta \mapsto \eta^\bullet$ between L^0 and $\mathbb{R}_{\mathcal{A}}\!\downarrow$. Let $\overline{\mathbb{R}}_{\mathcal{A}}$ denote the interpretation of the extended real numbers. Then, we can extend the canonical bijection to a bijection from $\bar{L}^0$ to $\overline{\mathbb{R}}_{\mathcal{A}}\!\downarrow$. Indeed, given $\eta \in \bar{L}^0$, we consider

$a := \{|\eta| < \infty\}$, $a_\infty := \{\eta = \infty\}$ and $a_{-\infty} := \{\eta = -\infty\}$. Then, by the mixing principle, there exists a unique $\eta^\bullet \in \overline{\mathbb{R}}_{\mathcal{A}}{\downarrow}$ such that $[\![\eta^\bullet \notin \{\pm\infty\}]\!] = a$, $[\![\eta^\bullet = \infty]\!] = a_\infty$ and $[\![\eta^\bullet = -\infty]\!] = a_{-\infty}$.

Next, we will list different relevant objects related to a given L^0-module E and will study the meaning of their ascents within $V^{(\mathcal{A})}$. Our purpose is to give the 'building blocks' for the module analogues of known statements of locally convex analysis, which will be also true due to the transfer principle:

2.3.1 Ascent and descent operations in an L^0-module

Let E be an L^0-module. We can consider $E{\upharpoonleft}$, which is a name for a vector space.

Given a stable subset S of E, we know that $S{\upharpoonleft}$ is a name for a non-empty subset of $E{\upharpoonleft}$. Given a stable collection $\mathscr{C}$ of stable subsets of E, then $\mathscr{C}{\Uparrow}$ is a name for a non-empty collection of non-empty subsets of $E{\upharpoonleft}$.

Next, we will define the inverse of the operations above.

Suppose that E is an L^0-module and let i_E the canonical isomorphism between E and $E{\upharpoonleft}{\downarrow}$:

- *Descent of a name for a subsets*: Given a name $\mathbb{S}$ for a non-empty subset of $E{\upharpoonleft}$ we define its descent to be $\mathbb{S}{\downharpoonleft} := i_E^{-1}(\mathbb{S}{\downarrow})$.
- *Descent of a name for a collection of subsets*: Given a name $\mathbb{v}$ for a non-empty collection of non-empty subsets of $E{\upharpoonleft}$ we define its descent to be

$$\mathbb{v}{\Downarrow} := \{\mathbb{S}{\downharpoonleft} \colon \mathbb{S} \in \mathbb{v}{\downarrow}\}.$$

Due to Proposition 2.2.10 and 2.2.11 we have that $\mathbb{S}{\downarrow}$ is a stable subset of $E{\upharpoonleft}{\downarrow}$ and $\mathbb{v}{\Downarrow}$ is a stable collection of stable subsets $E{\upharpoonleft}{\downarrow}$. Since i_E is an isomorphism of L^0-modules, we have that $\mathbb{S}{\downharpoonleft}$ is a stable subset of E and $\mathbb{v}{\Downarrow}$ is a stable collection of stable subsets of E.

Moreover, if S and stable subset of E, and $\mathscr{C}$ a stable collection of stable subsets of E, then $S{\upharpoonleft}{\downharpoonleft} = S$ and $\mathscr{C}{\Uparrow}{\Downarrow} = \mathscr{C}$.

In fact, the ascent operation $\cdot{\upharpoonleft}$ defines a bijection between the stable subsets of E and names for non-empty subsets of $E{\upharpoonleft}$, and the ascent operation $\Uparrow$ defines a bijection between stable collections of stable subsets of E and names for non-empty collections of non-empty sets of $E{\upharpoonleft}$.

Proposition 2.3.1. *Let E be an L^0-module, S a stable subset of E and $x \in E$. Then*

$$[\![x^\bullet \in S{\upharpoonleft}]\!] = \bigvee_{y \in S} a_{x,y}.$$

Moreover, there exists $z \in S$ such that $[\![x^\bullet \in S{\upharpoonleft}]\!] = a_{x,z}$.

Proof. Indeed,

$$[\![x^\bullet \in S{\upharpoonleft}]\!] = \bigvee_{y \in S} [\![y^\bullet = x^\bullet]\!] = \bigvee_{y \in S} a_{x,y}.$$

Set $a := [\![x^\bullet \in S{\upharpoonleft}]\!]$. We can find $(a_k) \in p(a)$ and $(y_k) \subset S$ such that $a_k \leq a_{x,y_k}$ for all $k \in \mathbb{N}$. Consequently, due to Lemma 2.2.1, $1_{a_k} x = 1_{a_k} y$ for all $k \in \mathbb{N}$. Fix some $y \in S$. Then $z := 1_{a^c} y + \sum 1_{a_k} y_k$ satisfies $a_{x,z} = a$. $\square$

The following result is easy to check by means of Theorem 2.1.4:

Proposition 2.3.2. *Let E be an L^0-module and S a stable subset of E, then:*

1. *S is L^0-convex if and only if* $[\![S\!\uparrow \text{ is convex}]\!] = I$;
2. *S is L^0-absorbing if and only if* $[\![S\!\uparrow \text{ is absorbing}]\!] = I$;
3. *S is L^0-balanced if and only if* $[\![S\!\uparrow \text{ is balanced}]\!] = I$.

Proposition 2.3.3. *Let E be an L^0-module and $\mathscr{C}$ a stable collection of stable subsets of E. Then:*

1. $[\![(\bigcup \mathscr{C})\!\uparrow = \bigcup(\mathscr{C}\!\Uparrow)]\!] = I$;
2. $[\![(\bigcap \mathscr{C})\!\uparrow = \bigcap(\mathscr{C}\!\Uparrow)]\!] = I$.

Proof. Each of the assertions can be checked by using Theorem 2.1.4 and taking into account that $S \mapsto S\!\uparrow$ is a bijection between $\mathscr{C}$ and $\mathscr{C}\!\Uparrow\downarrow$. □

Proposition 2.3.4. *Let $(E,\mathscr{T})$ be a locally stable L^0-convex module and suppose that $\mathscr{U}$ is a stable neighborhood base of $0 \in E$ consisting of stable subsets. Let S be a stable subset of E. Then*

1. *S is open if and only if* $[\![S\!\uparrow \text{ is } \langle\mathscr{U}\!\Uparrow\rangle\text{-open}]\!] = I$;
2. *S is closed if and only if* $[\![S\!\uparrow \text{ is } \langle\mathscr{U}\!\Uparrow\rangle\text{-closed}]\!] = I$;
3. $[\![\overline{S}\!\uparrow = \overline{S\!\uparrow}]\!] = I$ *and* $\left[\!\!\left[\overset{o}{S}\!\uparrow = \overset{o}{\widehat{S\!\uparrow}}\right]\!\!\right] = I$.

Furthermore,

$$\langle\mathscr{U}\!\Uparrow\rangle_{\mathcal{A}}\!\Downarrow = \{O \in \mathscr{T} : O \text{ is stable}\}.$$

Proof. 1 and 2 are consequences of 3. Thus, we will prove 3.

Notice that due to Proposition 1.4.2 it makes sense to define $\overline{S}\!\uparrow$ and $\overset{o}{S}\!\uparrow$. Now, by Lemma 1.2.2 we have that $\overline{S} = \bigcap_{U\in\mathscr{U}}(S+U)$. Then,

$$\begin{array}{lll} x \in \overline{S} & \text{if and only if} & x \in S + U \text{ for all } U \in \mathscr{U} \\ & \text{if and only if} & [\![x^\bullet \in S\!\uparrow + U\!\uparrow]\!] = I \text{ for all } U \in \mathscr{U} \\ & \text{if and only if} & [\![x^\bullet \in S\!\uparrow + \mathbb{U}]\!] = I \text{ for all } \mathbb{U} \in \mathscr{U}\!\Uparrow\downarrow \\ & \text{if and only if} & [\![(\forall U \in \mathscr{U}\!\Uparrow)(x^\bullet \in S\!\uparrow + U)]\!] = I \\ & \text{if and only if} & [\![x^\bullet \in \overline{S\!\uparrow}]\!] = I \end{array}.$$

This proves the first equality of 3.

Now

$$\begin{array}{lll} x \in \overset{o}{S} & \text{if and only if} & \text{there exists } U \in \mathscr{U} \text{ such that } x + U \subset S \\ & \text{if and only if} & \text{there exists } U \in \mathscr{U} \text{ such that } [\![x^\bullet + U\!\uparrow \subset S\!\uparrow]\!] = I \\ & \text{if and only if} & \text{there exists } \mathbb{U} \in \mathscr{U}\!\Uparrow\downarrow \text{ such that } [\![x^\bullet + \mathbb{U} \subset S\!\uparrow]\!] = I \\ & \text{if and only if} & [\![(\exists U \in \mathscr{U}\!\Uparrow)(x^\bullet + U \subset S\!\uparrow)]\!] = I \\ & \text{if and only if} & \left[\!\!\left[x^\bullet \in \overset{o}{\widehat{S\!\uparrow}}\right]\!\!\right] = I. \end{array}$$

This proves the second equality of 3.

Finally, using 1, we have that

$$\begin{aligned} O \in \langle \mathscr{U}\Uparrow\rangle_{\mathcal{A}}\Downarrow \quad & \text{if and only if} \quad O \text{ is stable and } [\![O\uparrow \text{ is open}]\!] = I \\ & \text{if and only if} \quad O \text{ is stable and open.} \end{aligned}$$

□

2.3.2 Stable filter bases and stable compactness

We showed in Chapter 1 that any Hausdorff locally stable L^0-convex module is anti-compact (see Theorem 1.4.2). Notice that this means that conventional compactness is not interesting in L^0-convex analysis because does not allow us to establish any meaningful compactness theorem. Next, We will see the suitable substitute for compactness, which will bring a wide range of compactness results due to the Boolean-valued transfer principle.

Definition 2.3.1. *Let $(E, \mathscr{T})$ be a locally stable L^0-convex module and S a stable subset of E:*

- *A filter base $\mathscr{G}$ on S which is a stable collection of stable subsets of E is called a* stable filter base*.*
- *A stable subset S is said to be* stably compact *if any stable filter base on S has a cluster point in S w.r.t. $\mathscr{T}$.*
- *We say that S is* relatively stably compact *if $\overline{S}$ is stably compact.*

The following result shows the connection between stable filter bases and names for filter bases:

Proposition 2.3.5. *Let $(E, \mathscr{T})$ be a locally stable L^0-convex module and suppose that $\mathscr{U}$ is a stable neighborhood base of $0 \in E$ consisting of stable subsets. Let $\mathscr{G}$ be a stable collection of stable subsets of E. Then $\mathscr{G}$ is a stable filter base if and only if*

$$[\![\mathscr{G}\Uparrow \text{ is a filter base}]\!] = I.$$

Furthermore, x is a cluster point of $\mathscr{G}$ if and only if

$$[\![x^\bullet \text{ is a } \langle\mathscr{U}\Uparrow\rangle\text{-cluster point of } \mathscr{G}\Uparrow]\!] = I.$$

Proof. $\mathscr{G}\Uparrow$ is a name for a non-empty collection of non-empty subsets. Since $U \mapsto U\uparrow$ is a bijection between $\mathscr{G}$ and $\mathscr{G}\Uparrow\downarrow$, by means of Theorem 2.1.4, we get

$$[\![(\forall U, V \in \mathscr{G}\Uparrow)(\exists W \in \mathscr{G}\Uparrow)(W \subset U \cap V)]\!] = I. \tag{2.4}$$

if and only if

for all $U, V \in \mathscr{G}$ there exists $W \in \mathscr{G}$ such that $W \subset U \cap V$.

This means that $\mathscr{G}\Uparrow$ is a name for a filter base if and only $\mathscr{G}$ is a stable filter base.

Similarly, for a given $x \in E$, Theorem 2.1.4 allows us to show that

$$[\![(\forall U \in \mathscr{U}\Uparrow)(\forall V \in \mathscr{G}\Uparrow)((x^\bullet + U) \cap V \neq \emptyset)]\!] = I$$

if and only if

for all $U \in \mathscr{U}$ and $V \in \mathscr{G}$, $(x + U) \cap V \neq \emptyset$.

This means that $x^\bullet$ is a name for a cluster point of $\mathscr{G}\Uparrow$ if and only if x is a cluster point of $\mathscr{G}$. □

Recall the elementary fact that a subset K is compact if and only if every filter base on K has a cluster point in K. Then, the transfer principle applied to this fact together with the proposition above yield the following:

Corollary 2.3.1. *Let $(E, \mathscr{T})$ be a locally stable L^0-convex module, $\mathscr{U}$ a stable neighborhood base of $0 \in E$ consisting of stable subsets, and suppose that $S \subset E$ is stable. Then:*

1. *S is stably compact if and only if* $[\![S\!\uparrow \text{ is } \langle \mathscr{U}\!\Uparrow\rangle\text{-compact}]\!] = I$;
2. *relatively stably compact if and only if* $[\![S\!\uparrow \text{ is relatively } \langle \mathscr{U}\!\Uparrow\rangle\text{-compact}]\!] = I$.

Comments 2.3.1. *Stable compactness was first time studied by Kusraev [74] in the case of descents of normed spaces giving rise to the notion of* cyclic compactness. *More recently, the notion of* mix-compactness *was introduced by Gutman and Lisovskaya [58]. It turns out that* cyclic compactness *and* mix-compactness *are equivalent notions (see [80, Theorem 2.12.C.5]). These types of compactness have been fruitfully exploited, see for instance results in [76, Sections 1.3 and 1.4], [77, Section 8.5] and the analogues of the boundedness and uniform boundedness principles obtained in [58].*

2.3.3 Stable functions

Definition 2.3.2. *Let E_1 and E_2 be L^0-modules and suppose that S_1 and S_2 are stable subsets of E_1 and E_2, respectively, and $\mathscr{C}_1$ and $\mathscr{C}_2$ are stable collections of stable subsets of E_1 and E_2, respectively. A function $f : X \to Y$ with*

$$X \in \{S_1, \mathscr{C}_1\} \quad \textit{and} \quad Y \in \{S_2, \mathscr{C}_2, \bar{L}^0\},$$

is said to be stable *if*

$$f(\sum_k 1_{a_k} x_k) = \sum_k 1_{a_k} f(x_k) \quad \textit{whenever } (x_k) \subset X \textit{ and } (a_k) \in p(I).$$

The following result defines ascents operations for different types of stable functions:

Proposition 2.3.6. *Let E_1, E_2 be L^0-modules, S_1 and S_2 stable subsets of E_1 and E_2, respectively, and $\mathscr{C}_1$ and $\mathscr{C}_2$ stable collections of stable subsets of E_1 and E_2, respectively. Then:*

1. *If $f : S_1 \to \bar{L}^0$ is stable, then there exists a name $f\!\uparrow$ such that*
$$[\![f \in \mathrm{fun}(S_1\!\uparrow, \overline{\mathbb{R}}_{\mathcal{A}})]\!] = I \quad \textit{and} \quad [\![f\!\uparrow(x^\bullet) = f(x)^\bullet]\!] = I \text{ for all } x \in S_1.$$
2. *If $f : S_1 \to S_2$ is stable, then there exists a name $f\!\uparrow$ such that*
$$[\![f \in \mathrm{fun}(S_1\!\uparrow, S_2\!\uparrow)]\!] = I \quad \textit{and} \quad [\![f\!\uparrow(x^\bullet) = f(x)^\bullet]\!] = I \text{ for all } x \in S_1.$$
3. *If $f : S_1 \to \mathscr{C}_2$ is stable, then there exists a name $f\!\uparrow$ such that*
$$[\![f \in \mathrm{fun}(S_1\!\uparrow, \mathscr{C}_2\!\Uparrow)]\!] = I \quad \textit{and} \quad [\![f\!\uparrow(x^\bullet) = f(x)\!\uparrow]\!] = I \text{ for all } x \in S_1.$$
4. *If $f : \mathscr{C}_1 \to S_2$ is stable, then there exists a name $f\!\uparrow$ such that*
$$[\![f \in \mathrm{fun}(\mathscr{C}_1\!\Uparrow, S_2\!\uparrow)]\!] = I \quad \textit{and} \quad [\![f\!\uparrow(S\!\uparrow) = f(S)^\bullet]\!] = I \text{ for all } S \in \mathscr{C}_1.$$
5. *If $f : \mathscr{C}_1 \to \mathscr{C}_2$ is stable, then there exists a name $f\!\uparrow$ such that*
$$[\![f \in \mathrm{fun}(\mathscr{C}_1\!\Uparrow, \mathscr{C}_2\!\Uparrow)]\!] = I \quad \textit{and} \quad [\![f\!\uparrow(S\!\uparrow) = f(S)\!\uparrow]\!] = I \text{ for all } S \in \mathscr{C}_1.$$

Proof. Let us prove 1. Consider the function

$$f_0 : S_1{\uparrow}{\downarrow} \longrightarrow \overline{\mathbb{R}}_{\mathcal{A}} \quad x^\bullet \mapsto f(x)^\bullet.$$

We claim that f_0 is extensional. Indeed, take $x, y \in S_1$. Fix $z_0 \in S_1$. Since f is stable

$$\begin{aligned} 1_{a_{x,y}} f(x) &= 1_{a_{x,y}} f(1_{a_{x,y}} x + 1_{a^c_{x,y}} x) \\ &= 1_{a_{x,y}} f(1_{a_{x,y}} y + 1_{a^c_{x,y}} x) \\ &= 1_{a_{x,y}} (1_{a_{x,y}} f(y) + 1_{a^c_{x,y}} f(x)) = 1_{a_{x,y}} f(y). \end{aligned}$$

Then $[\![x^\bullet = y^\bullet]\!] = a_{x,y} \leq a_{f(x),f(y)} = [\![f(x)^\bullet = f(y)^\bullet]\!]$, as desired. Thereby, Theorem 2.1.5 yields a name $f{\uparrow}$ for a function which satisfies the conditions.

In the remaining cases, we can define extensional functions between the suitable descents. Then Theorem 2.1.5 provides us with the desired names. □

For all these ascent operations for stable functions, it can be defined the corresponding inverse operation of descent. For instance, suppose that E_1 and E_2 are L^0-modules and $\mathbb{f}$ is a name for a function from $E_1{\uparrow}$ to $E_2{\uparrow}$. We define its descent by $\mathbb{f}{\downarrow} := i_{E_2}^{-1} \circ \mathbb{f}{\downarrow} \circ i_{E_1}$, which is a stable function.

Recall that, if $\lambda : X \to \overline{\mathbb{R}}$ is a function defined on a vector space X, its *efective domain* is

$$\operatorname{dom}(\lambda) := \{x \in X \colon \lambda(x) \in \mathbb{R}\},$$

and λ is said to be *proper* if $\lambda(x) < \infty$ for at least one $x \in X$ and $f(x) > -\infty$ for all $x \in X$.

The following nomenclature comes from L^0-convex analysis (see [36]):

Definition 2.3.3. *Suppose that $f : E \to \bar{L}^0$ is a function defined on an L^0-module E.*

- *f has the* local property *if*

$$1_a f(x) = 1_a f(1_a x) \qquad \text{for all } a \in \mathcal{A} \text{ and } x \in E;$$

- *f is L^0-*convex *if $f(\eta x + (1-\eta)y) \leq \eta f(x) + (1-\eta) f(y)$ for all $x, y \in E$ and $\eta \in L^0$ with $0 \leq \eta \leq 1$;*
- *the* Effective domain *of f is defined to be* $\operatorname{dom}(f) := \{x \in E \colon f(x) \in L^0\}$.
- *The function f is said to be* proper *if $f(x) > -\infty$ for all $x \in E$ and there exists some $x \in \operatorname{dom}(f)$.*

Proposition 2.3.7. *Let E be an L^0-module and $f : E \to \bar{L}^0$ a function. Then f has the local property if, and only if, f is stable. In particular, there is a name $f{\uparrow}$ for a function from $E{\uparrow}$ to $\overline{\mathbb{R}}_{\mathcal{A}}$ such that*

$$[\![f{\uparrow}(x^\bullet) = f(x)^\bullet]\!] = I \text{ for all } x \in E.$$

Furthermore, $[\![\operatorname{dom}(f){\uparrow} = \operatorname{dom}(f{\uparrow})]\!] = I$ *and f is proper if and only if* $[\![f{\uparrow} \text{ is proper}]\!] = I$.

Proof. Suppose that f has the local property. Take $(a_k) \in p(I)$ and $(x_k) \subset E$. Set $x = \sum 1_{a_k} x_k$. Then, for each $k \in \mathbb{N}$

$$1_{a_k} f(x) = 1_{a_k} f(1_{a_k} x) = 1_{a_k} f(1_{a_k} x_k) = 1_{a_k} f(x_k).$$

This proves that f is stable.

Conversely, suppose that f is stable. Take $a \in \mathcal{A}$ and $x \in E$. Then

$$1_a f(x) = 1_a f(1_a(1_a x) + 1_{a^c} x) = 1_a(1_a f(1_a x) + 1_{a^c} f(x)) = 1_a f(1_a x).$$

It follows that f has that local property. The rest of assertions can be easily checked by using Theorem 2.1.4. □

It is well-known that any L^0-convex function $f : E \to \bar{L}^0$ has the local property and it is therefore stable (see eg [36, Theorem 3.2]). Thus the following holds:

Proposition 2.3.8. *If $f : E \to \bar{L}^0$ is L^0-convex, then f is stable and $f\!\uparrow$ is a name for a convex function.*

An identical argument as that that proves Proposition 2.2.15 shows the following:

Proposition 2.3.9. *Let $(E_1, \mathscr{T}_1)$, $(E_2, \mathscr{T}_2)$ be locally stable L^0-modules, and $\mathscr{U}_1$ and $\mathscr{U}_2$ stable neighborhood bases of $0 \in E_1$ and $0 \in E_2$ consisting of stable subsets, respectively. If $f : E_1 \to E_2$ has the local property, then f is continuous if and only if $f\!\uparrow$ is a name for a continuous function with respect the topologies $\langle \mathscr{U}_1 \Uparrow \rangle_{\mathcal{A}}$ and $\langle \mathscr{U}_2 \Uparrow \rangle_{\mathcal{A}}$.*

The fact that $x \mapsto x^\bullet$ is a bijection between S and $S\!\uparrow\downarrow$ together with Theorem 2.1.4 allow us to prove the following:

Proposition 2.3.10. *If S is a stable subset of E and $f : S \to \bar{L}^0$ is a stable function, one has*

$$\left[\!\!\left[\sup_{x \in S\uparrow} f\!\uparrow(x) = (\sup_{z \in S} f(z))^\bullet \right]\!\!\right] = I \quad \textit{and} \quad \left[\!\!\left[\inf_{x \in S\uparrow} f\!\uparrow(x) = (\inf_{z \in S} f(z))^\bullet \right]\!\!\right] = I.$$

Recall that a function $\lambda : X \to \overline{\mathbb{R}}$ defined on a locally convex space (X, τ) is said to be *lower semi-continuous* if the r-sub-level set $V_r(\lambda) := \{x \colon \lambda(x) \leq r\}$ is closed for all $r \in \mathbb{R}$; it is said to be *inf-compact* if $V_r(\lambda)$ is compact for all $r \in \mathbb{R}$.

These notions are generalized to the setting of L^0-modules as follows:

Definition 2.3.4. *Given a function $f : E \to \bar{L}^0$, we define its η-sub-level set ($\eta \in L^0$) to be*

$$V_\eta(f) = \{x \in E \colon f(x) \leq \eta\}.$$

Definition 2.3.5. *A function $f : E \to \bar{L}^0$ defined on a locally L^0-convex module $(E, \mathscr{T})$ is said to be:*

- lower semi-continuous*, l.s.c. for short, if $V_\eta(f)$ is closed for every $\eta \in L^0$;*
- stably inf-compact *if for each $\eta \in L^0$, $V_\eta(f)$ is either stably compact or empty.*

We have the following:

Proposition 2.3.11. *Let $(E, \mathscr{T})$ be a locally stable L^0-convex module and suppose that $\mathscr{U}$ is a stable base neighborhood of $0 \in E$ consisting of stable subsets and suppose that $f : E \to \bar{L}^0$ is proper and has the local property. We have the following:*

1. *f is lower semi-continuous w.r.t. $\mathscr{T}$ if, and only if,*

$$[\![f \text{ is lower semi-continuous w.r.t. } \langle \mathscr{U} \Uparrow \rangle]\!] = I.$$

2. f is stably inf-compact w.r.t. $\mathscr{T}$ if, and only if,

$$[\![f \text{ is inf-compact w.r.t. } \langle \mathscr{U}\Uparrow \rangle]\!] = I.$$

Proof. Let us prove 1. Notice that, given $\eta \in L^0$, if $V_\eta(f) \neq \emptyset$, then $V_\eta(f)$ is a stable set due to the stability of f. Consequently, we can consider $V_\eta(f)\uparrow$ and it satisfies $[\![V_\eta(f)\uparrow = V_{\eta^\bullet}(f\uparrow)]\!] = I$.

Suppose that $[\![f\uparrow \text{ is l.s.c.}]\!] = \Omega$. Take $\eta \in L^0$. If $V_\eta(f) = \emptyset$, then it is closed. If $V_\eta(f) \neq \emptyset$, then it is closed as $[\![V_\eta(f)\uparrow \text{ is closed}]\!] = \Omega$.

Conversely, suppose that f is l.s.c.. Since f is proper, we can fix $\xi \in L^0$ such that $V_\xi(f) \neq \emptyset$. For each $\eta \in L^0$, define $a_\eta := [\![V_\eta(f)\uparrow \neq \emptyset]\!]$ and $\xi_\eta = 1_{a_\eta}\eta + 1_{a_\eta^c}\xi$. Then

$$\begin{aligned}
[\![f\uparrow \text{ is l.s.c. }]\!] &= [\![(\forall r \in \mathbb{R})(V_r(f\uparrow) \text{ is closed})]\!] \\
&= \bigwedge_{\eta \in L^0} [\![V_\eta(f)\uparrow \text{ is closed}]\!] \\
&= \bigwedge_{\eta \in L^0} [\![V_\eta(f)\uparrow = \emptyset]\!] \vee [\![(V_\eta(f)\uparrow \neq \emptyset) \wedge (V_\eta(f)\uparrow \text{ is closed})]\!] \\
&\geq \bigwedge_{\eta \in L^0} [\![V_\eta(f)\uparrow = \emptyset]\!] \vee [\![(V_\eta(f)\uparrow \neq \emptyset) \wedge (\eta^\bullet = \xi_\eta^\bullet) \wedge (V_{\xi_\eta}(f)\uparrow \text{ is closed})]\!] \\
&= \bigwedge_{\eta \in L^0} a_\eta^c \vee (a_\eta \wedge [\![\eta^\bullet = \xi_\eta^\bullet]\!] \wedge [\![V_{\xi_\eta}(f)\uparrow \text{ is closed}]\!]) \\
&\geq \bigwedge_{\eta \in L^0} a_\eta^c \vee (a_\eta \wedge a_\eta \wedge I) = I.
\end{aligned}$$

Similarly, one proves 2 taking into account that $V_\eta(f)$ is stably compact if and only if $[\![V_\eta(f)\uparrow \text{ is compact}]\!] = I$. □

Definition 2.3.6. *Let E_1 and E_2 be L^0-modules and suppose that S_1 and S_2 are stable subsets of E_1 and E_2, respectively, and $\mathscr{C}_1$ and $\mathscr{C}_2$ are stable collections of stable subsets of E_1 and E_2, respectively. Suppose that $X \in \{S_1, \mathscr{C}_1\}$ and $Y \in \{S_2, \mathscr{C}_2, \bar{L}^0\}$. Then a family $(y_x)_{x \in X}$ of elements of Y is said to be* stable *if the function $x \mapsto y_x$ is stable*

The transfer principle applied to the axiom of choice proves the following version of the axiom of choice, which can be also derived from the conditional version of the axiom of choice established in [32]

Theorem 2.3.1. *Let E_1, E_2 be L^0-modules, S_1 a stable subset of E_1, and $\mathscr{C}_1$ and $\mathscr{C}_2$ stable collections of stable subsets of E_1 and E_2, respectively. Suppose that $(L_x)_{x \in X}$ is a stable family of elements of $\mathscr{C}_2$ with $X \in \{S_1, \mathscr{C}_1\}$, then there exists a stable family $(z_x)_{x \in X}$ of elements of E_1 such that $z_x \in L_x$ for all $x \in X$.*

Definition 2.3.7. *Let E be an L^0-module. A stable family $(x_\mathfrak{n})_{\mathfrak{n} \in L^0(\mathbb{N})}$ in E is called a* stable sequence. *A stable sequence $(y_\mathfrak{n})_{\mathfrak{n} \in L^0(\mathbb{N})}$ in E is said to be a* stable subsequence of $(x_\mathfrak{n})_{\mathfrak{n} \in L^0(\mathbb{N})}$ *if there exists a stable sequence $(\mathfrak{n}_\mathfrak{m})_{\mathfrak{m} \in L^0(\mathbb{N})}$ in $L^0(\mathbb{N})$ such that $\mathfrak{n}_\mathfrak{m} < \mathfrak{n}_{\mathfrak{m}'}$ whenever $\mathfrak{m} < \mathfrak{m}'$ and $y_\mathfrak{m} = x_{\mathfrak{n}_\mathfrak{m}}$ for all $\mathfrak{m} \in L^0(\mathbb{N})$.*

Proposition 2.3.12. *Let E be an L^0-module. The following holds:*

1. *If $\chi := (x_\mathfrak{n})_{\mathfrak{n} \in L^0(\mathbb{N})}$ is a stable sequence then $\chi\uparrow$ is a name for a sequence;*
2. *If $\chi := (x_\mathfrak{n})_{\mathfrak{n} \in L^0(\mathbb{N})}$ is a stable sequence and $\chi' := (x_{\mathfrak{n}_\mathfrak{m}})_{\mathfrak{m} \in L^0(\mathbb{N})}$ is a stable subsequence of χ, then $\chi'\uparrow$ is a name for a subsequence of $\chi\uparrow$.*

Proposition 2.3.13. *Let $(E,\mathscr{T})$ be a locally stable L^0-convex module with a stable neighborhood base $\mathscr{U}$ of $0\in E$ consisting of stable subsets. Suppose that $(x_{\mathfrak{n}})_{\mathfrak{n}\in L^0(\mathbb{N})}$ is a stable sequence in E, then*

$$\lim_{\mathfrak{n}} x_{\mathfrak{n}} = x \textit{ w.r.t. } \mathscr{T} \quad \textit{if and only if} \quad [\![\lim_n x^{\bullet}_{n^\circ} = x^\bullet \text{ w.r.t. } \langle \mathscr{U}\Uparrow\rangle]\!] = I.$$

Proof. $[\![\lim_n x^\bullet_{n^\circ} = x^\bullet]\!] = I$ means that $[\![(\forall U\in\mathscr{U}\Uparrow)(\exists n\in\mathbb{N})(\forall m\geq n)(x^\bullet_{m^\circ}\in x^\bullet+U)]\!]=I$. Then

for all $U\in\mathscr{U}$ there exists $\mathfrak{n}_U\in L^0(\mathbb{N})$ such that $x_{\mathfrak{n}}\in U$ whenever $\mathfrak{n}\geq\mathfrak{n}_U$.

Conversely, suppose that $\lim_{\mathfrak{n}} x_{\mathfrak{n}} = x$. Fixed $U\in\mathscr{U}$ we define

$$S_U := \{\mathfrak{m}\in L^0(\mathbb{N})\colon x_{\mathfrak{n}}\in x+U \text{ for all } \mathfrak{n}\geq\mathfrak{m}\}.$$

Then, the family $(S_U)_{U\in\mathscr{U}}$ is stable. By Theorem 2.3.1, we can find a stable family $(\mathfrak{n}_U)_{U\in\mathscr{U}}$ with $\mathfrak{n}_U\in S_U$ for all $U\in\mathscr{U}$. Thus we can consider the name $(\mathfrak{n}_U)_{U\in\mathscr{U}}\uparrow$, which verifies

$$[\![(\forall m\geq\mathfrak{n}^\bullet_U)(x^\bullet_{m^\circ}\in x^\bullet+U\uparrow)]\!]=I \quad \text{for all } U\in\mathscr{U},$$

and the assertion follows. □

Given $\mathfrak{m}\in L^0(\mathbb{N})$, we define $L^0_{\mathfrak{m}}(\mathbb{N}) := \{\mathfrak{n}\in L^0(\mathbb{N})\colon \mathfrak{n}\leq\mathfrak{m}\}$.

Definition 2.3.8. *Let E be an L^0-module and S a stable subset of E.*

- *Say that S is* stably countable *if there exists a stable surjection $\mathfrak{n}\mapsto x_{\mathfrak{n}}$ from $L^0(\mathbb{N})$ to S;*
- *say that S is* stably finite *if there exist $\mathfrak{m}\in L^0(\mathbb{N})$ and a stable surjection $\mathfrak{n}\mapsto x_{\mathfrak{n}}$ from $L^0_{\mathfrak{m}}(\mathbb{N})$ to S.*

We have the following:

Proposition 2.3.14. *Let S be a stable subset of an L^0-modules E, then:*

1. *S is stably finite if and only if $[\![S\uparrow$ is finite$]\!]=I$;*
2. *S is stably countable if and only if $[\![S\uparrow$ is countable$]\!]=I$.*

Proof. Suppose that S is stable finite, that is, $S=\{x_{\mathfrak{n}}\colon \mathfrak{n}\in L^0_{\mathfrak{m}}(\mathbb{N})\}$ for $\mathfrak{m}\in L^0(\mathbb{N})$ and a stable family $(x_{\mathfrak{n}})_{\mathfrak{n}\in L^0_{\mathfrak{m}}(\mathbb{N})}$. Then $[\![S\uparrow=\{x^\bullet_{n^\circ}\colon n\in\mathbb{N},\, n\leq\mathfrak{m}^\bullet\}]\!]=I$. We conclude that $S\uparrow$ is a name for a finite set.

Conversely, if $S\uparrow$ is a name for a finite set, then we can find names $\mathbb{m}$ and $\mathbb{x}$ for a natural number and for a finite sequence in $S\uparrow$, respectively, such that $[\![S\uparrow=\{\mathbb{x}_n\colon n\leq\mathbb{m}\}]\!]=I$. Set $\mathfrak{m}:=\mathbb{m}^\circ$ and, for each $\mathfrak{n}\in L^0_{\mathfrak{m}}(\mathbb{N})$, take $x_{\mathfrak{n}}\in S$ with $x^\bullet_{\mathfrak{n}}=\mathbb{x}_{\mathfrak{n}^\bullet}$. Then $S=\{x_{\mathfrak{n}}\colon\mathfrak{n}\in L^0_{\mathfrak{m}}(\mathbb{N})\}$, hence S is stably finite.

A similar argument proves 2. □

2.3.4 Stably finitely generated L^0-modules

Let E be an L^0-module:

- Given a non-empty subset S of E we define its *stable L^0-span* to be

$$\operatorname{span}^{\mathfrak{s}}_{L^0}(S) := \bigcap\{F\colon F \text{ is a stable } L^0\text{-submodule of } E \text{ with } S\subset F\}.$$

  Notice that $\operatorname{span}^{\mathfrak{s}}_{L^0}(S)$ is the least stable L^0-submodule that contains S.

- Say that E is *stably finitely generated* if there exists a stably finite subset S of E such that
$$E = \mathrm{span}^{\mathfrak{s}}_{L^0}(S).$$

An easy manipulation of Boolean truth values using Theorem 2.1.4 proves the following:

Proposition 2.3.15. *Let E be an L^0-module and S a stable subset of E. The following is satisfied:*

1. $[\![(\mathrm{span}^{\mathfrak{s}}_{L^0}(S))\!\uparrow = \mathrm{span}_{\mathbb{R}}(S\!\uparrow)]\!] = I$;
2. *E is stably finitely generated if and only if* $[\![E\!\uparrow \text{ is finitaly generated}]\!] = I$.

2.3.5 L^0-normed modules and stable completeness

If $\|\cdot\| : E \to L^0_+$ is an L^0-norm defined on an L^0-module E, we saw in Chapter 1 that it induces a Hausdorff locally stable L^0-convex topology on E.

Definition 2.3.9. *Let $(E, \|\cdot\|)$ be an L^0-normed module.*

- *A stable sequence $(x_{\mathfrak{n}})_{\mathfrak{n}\in L^0(\mathbb{N})}$ is said to be* Cauchy *if for every $\varepsilon \in L^0_{++}$ there exists $\mathfrak{n}_\varepsilon \in L^0(\mathbb{N})$ such that*
$$\|x_{\mathfrak{n}} - x_{\mathfrak{m}}\| \le \varepsilon \quad \textit{whenever } \mathfrak{n}, \mathfrak{m} \in L^0(\mathbb{N}) \textit{ and } \mathfrak{n}, \mathfrak{m} \ge \mathfrak{n}_\varepsilon.$$
- *E is said to be* stably complete *if every Cauchy stable sequence in E converges in E.*

The following result exhibits how the ascent of an L^0-normed module is connected to names of normed spaces:

Proposition 2.3.16. *Let $(E, \|\cdot\|)$ be an L^0-normed module. Then $(E\!\uparrow, \|\cdot\|\!\uparrow)_{\mathcal{A}}$ is a name for a normed vector space. Moreover:*

1. *A stable sequence $(x_{\mathfrak{n}})_{\mathfrak{n}\in L^0(\mathbb{N})} \subset E$ is Cauchy if and only if* $[\![(x^\bullet_{n^\circ})_{n\in\mathbb{N}} \text{ is Cauchy}]\!] = I$.
2. *$(E, \|\cdot\|)$ is stably complete if and only if $(E\!\uparrow, \|\cdot\|\!\uparrow)_{\mathcal{A}}$ is a name for a Banach space.*

Proof. The function $\|\cdot\| : E \to L^0_+$ has the local property, thus we can define $\|\cdot\|\!\uparrow$ which is clearly a name for a norm.

Let $(x_{\mathfrak{n}})_{\mathfrak{n}\in L^0(\mathbb{N})} \subset E$ be a stable sequence. If $[\![(x^\bullet_{n^\circ})_{n\in\mathbb{N}} \text{ is Cauchy}]\!] = I$, then it is clear that $(x_{\mathfrak{n}})_{\mathfrak{n}\in L^0(\mathbb{N})}$ is Cauchy.

Conversely, if $(x_{\mathfrak{n}})_{\mathfrak{n}\in L^0(\mathbb{N})}$ is Cauchy, for any $\varepsilon \in L^0_{++}$ set
$$S_\varepsilon := \{\mathfrak{n} \in L^0(\mathbb{N})\colon \|x_{\mathfrak{p}} - x_{\mathfrak{q}}\| \le \varepsilon,\ \forall \mathfrak{p}, \mathfrak{q} \ge \mathfrak{n}\}.$$
S_ε is non-empty for each ε. In addition, $(S_\varepsilon)_{\varepsilon\in L^0_{++}}$ is a stable family. Then Theorem 2.3.1 allows us to pick up a stable family $(\mathfrak{n}_\varepsilon)_{\varepsilon\in L^0_{++}}$ such that $\mathfrak{n}_\varepsilon \in S_\varepsilon$ for each $\varepsilon \in L^0_{++}$. Then, we have that
$$[\![(\forall r > 0)(\forall p, q \ge \mathfrak{n}^\bullet_{r^\circ})(|x^\bullet_{p^\circ} - x^\bullet_{q^\circ}| \le r)]\!] = I,$$
and the conclusion follows.

Finally, the correspondence between Cauchy stable sequences and names for Cauchy sequences proves that $(E, \|\cdot\|)$ is stably complete if and only if $(E\!\uparrow, \|\cdot\|\!\uparrow)_{\mathcal{A}}$ is a name for a Banach space. □

Comments 2.3.2. *Descents of Banach spaces were studied earlier by Kusraev [78], originating the notion of Banach-Kantorovich space, which are descents of real Banach spaces as proven in [78] (for further details see [77, Section 8.3] and [81, Section 5.4]).*

2.3.6 Dual L^0-module

In the following, we will define the *ascent of the dual L^0-module* of a locally stable L^0-convex module.

Let $(E, \mathscr{T})$ be a locally stable L^0-convex module. We define the ascent of its dual L^0-module $E^*[\mathscr{T}]$, denoted by $E^*[\mathscr{T}]\uparrow$, or simply $E^*\uparrow$, the unique representative in $\overline{V}^{(\mathcal{A})}$ of the name given by

$$\{\mu\uparrow\colon \mu \in E^*\} \longrightarrow \mathcal{A} \quad \mu\uparrow \mapsto I.$$

Remark 2.3.1. *Notice that the record $E^*\uparrow$ denotes also the ascent of E^* as an L^0-module, whose formal definition is different than its ascent as a dual L^0-module. This will not arise risk of confusion as we will always assume that $E^*\uparrow$ unequivocally denotes the ascent of the dual L^0-module of E.*

The following result is clear:

Proposition 2.3.17. *Let $(E, \mathscr{T})$ be a locally stable L^0-convex module and $\mathscr{U}$ a stable neighborhood of $0 \in E$ consisting of stable neighborhoods. Then $E^*[\mathscr{T}]\uparrow$ is a name for the dual space of $(E\uparrow, \langle \mathscr{U}\Uparrow\rangle)_{\mathcal{A}}$, that is, $[\![E^*[\mathscr{T}]\uparrow = (E\uparrow)^*[\langle \mathscr{U}\Uparrow\rangle]]\!] = I$.*

Recall that, given a locally convex space (X, τ), the Fenchel conjugate of a function $\lambda : X \to \overline{\mathbb{R}}$ is

$$\lambda^*(\theta) := \sup_{x\in X}(\theta(x) - \lambda(x)) \quad \text{ for } \theta \in X^*,$$

and its biconjugate is

$$\lambda^{**}(x) := \sup_{\theta\in X^*}(\theta(x) - \lambda^*(\theta)) \quad \text{ for } x \in X,$$

where $X^* := X^*[\tau]$ denotes the topological dual of X.

An element $\theta \in X^*[\tau]$ is a subgradient of $\lambda : X \to \overline{R}$ at $x_0 \in \operatorname{dom}(\lambda)$ if

$$\theta(x - x_0) \le \lambda(x) - \lambda(x_0) \quad \text{ for all } x \in X.$$

Filipovic et al. [36] introduced module analogues of the classical Fenchel conjugate and the notion of subgradient. Namely, if $(E, \mathscr{T})$ is a locally L^0-convex module, the *conjugate* of a function $f : E \to \bar{L}^0$ is defined by

$$f^* : E^* \to \bar{L}^0, \quad f^*(\mu) := \sup_{x\in E}(\mu(x) - f(x)),$$

and its *biconjugate* is defined by

$$f^{**} : E \to \bar{L}^0, \quad f^{**}(x) := \sup_{\mu\in E^*}(\mu(x) - f^*(\mu)).$$

An element $\mu \in E^*[\mathscr{T}]$ is a subgradient of $f : E \to \bar{L}^0$ at $x_0 \in \operatorname{dom}(f)$ if

$$\mu(x - x_0) \le f(x) - f(x_0) \quad \text{ for all } x \in E.$$

The record $\partial f(x_0)$ stands for the set of all subgradients of f at x_0.

Proposition 2.3.18. *Let $(E, \mathscr{T})$ be a locally stable L^0-convex module and let $f : E \to \bar{L}^0$ be a function with the local property. Then it is satisfied:*

1. $[\![(f^*)\uparrow = (f\uparrow)^*]\!] = I$;

2. $[\![(f^{**})\!\uparrow = (f\!\uparrow)^{**}]\!] = I$.

In addition, if $\mu \in E^*$*, then* $\mu \in \partial f(x_0)$ *if and only if* $[\![\mu\!\uparrow \in \partial f\!\uparrow(x_0^\bullet)]\!] = I$.

Proof. 1. Given $\mu \in E^*$, due to Proposition 2.3.10, one has

$$\left[\!\!\left[\sup_{x \in E\uparrow} (\mu\!\uparrow\!(x) - f\!\uparrow\!(x)) = (\sup_{z \in E}(\mu(z) - f(x)))^\bullet\right]\!\!\right] = I.$$

This means that

$$[\![(f\!\uparrow)^*(\mu\!\uparrow) = (f^*(\mu))^\bullet]\!] = I,$$

and thus $[\![(f\!\uparrow)^*(\mu\!\uparrow) = (f^*)\!\uparrow\!(\mu\!\uparrow)]\!] = I$. Since $\mu \mapsto \mu\!\uparrow$ is a bijection between E^* and $E^*\!\downarrow$, the assertion follows.

2. Given $x \in E$, again due to Proposition 2.3.10, one has

$$\left[\!\!\left[\sup_{\theta \in (E^*)\uparrow} (\theta(x^\bullet) - (f^*)\!\uparrow\!(\theta)) = (\sup_{\mu \in E^*}(\mu(x) - f^*(\mu)))^\bullet\right]\!\!\right] = I.$$

Since $[\![(E^*)\!\uparrow = (E\!\uparrow)^*]\!] = I$ and $[\![(f^*)\!\uparrow = (f\!\uparrow)^*]\!] = I$, one has

$$[\![(f\!\uparrow)^{**}(x^\bullet) = f^{**}(x^\bullet)]\!] = I$$

and the assertion follows.

□

Comments 2.3.3. *Earlier than [36], conjugates and subgradients of functions taking values in universaly complete vector lattices have been studied in Boolean-valued analysis (see eg [79, Chapter 4]).*

2.3.7 Dual systems of L^0-modules

Definition 2.3.10. *Let* E_1, E_2 *be* L^0*-modules. The pair* (E_1, E_2) *is called a* dual system of L^0-modules *with respect the bi-*L^0*-module homomorphism*

$$\langle \cdot, \cdot \rangle : E_1 \times E_2 \longrightarrow L^0$$

(i.e. both $\langle \cdot, y \rangle : E_1 \to L^0$ *and* $\langle x, \cdot \rangle : E_2 \to L^0$ *are* L^0*-module homomorphisms) if the following conditions are fulfilled:*

1. $\langle x, y \rangle = 0$ *for all* $x \in E_1$ *implies* $y = 0$*;*
2. $\langle x, y \rangle = 0$ *for all* $y \in E_2$ *implies* $x = 0$*.*

For the sake of brevity, we will say that $\langle E_1, E_2 \rangle$ *is a dual system of* L^0*-modules.*

The following proposition defines the ascent of a dual system of L^0-modules.

Proposition 2.3.19. *Let* $\langle E_1, E_2 \rangle$ *be a dual system of* L^0*-modules. Then there exists a name* $\langle E_1\!\uparrow, E_2\!\uparrow \rangle_{\mathcal{A}}$ *for a dual pair such that*

$$[\![\langle x^\bullet, y^\bullet \rangle = \langle x, y \rangle^\bullet]\!] = I \quad \text{for all } x \in E_1,\ y \in E_2.$$

Proof. The function

$$E_1{\Uparrow}\!\downarrow \times E_2{\Uparrow}\!\downarrow \longrightarrow \mathbb{R}_{\mathcal{A}} \quad (x^\bullet, y^\bullet) \mapsto \langle x, y\rangle^\bullet,$$

is extensional; due to Theorem 2.1.6 there exists a name $\langle\cdot,\cdot\rangle{\uparrow}$ for a bi-linear function from $(E_1{\uparrow} \times E_2{\uparrow})_{\mathcal{A}}$ to $\mathbb{R}_{\mathcal{A}}$ such that

$$[\![\langle x^\bullet, y^\bullet\rangle{\uparrow} = \langle x, y\rangle^\bullet]\!] = I \quad \text{ for all } x \in E_1,\ y \in E_2.$$

Moreover, Theorem 2.1.4 allows us to prove that $\langle\cdot,\cdot\rangle{\uparrow}$

$$[\![(\forall y \in E_2)\,(((\forall x \in E_1{\uparrow})\langle x, y\rangle = 0) \Rightarrow y = 0)]\!] = I,$$

$$[\![(\forall x \in E_1)\,(((\forall y \in E_2{\uparrow})\langle x, y\rangle = 0) \Rightarrow x = 0)]\!] = I.$$

Thus we conclude that $\langle E_1{\uparrow}, E_2{\uparrow}\rangle_{\mathcal{A}}$ is a name for a dual pair. □

Given a dual system of L^0-modules $\langle E_1, E_2\rangle$, for any $y \in E_2$, we can define

$$\|\cdot\|_y : E_1 \longrightarrow L^0_+ \quad \|x\|_y := |\langle x, y\rangle|.$$

Then $\{\|\cdot\|_y : y \in E_2\}$ is a family of L^0-seminorms which stably induces the stably weak topology $\sigma_{\mathfrak{s}}(E_1, E_2)$.

Let $\mathscr{U}_{\langle E_1,E_2\rangle}$ denote a stable neighborhood base of $0 \in E_1$ consisting of stable subsets for the topology $\sigma_{\mathfrak{s}}(E_1, E_2)$. Then we have the following:

Proposition 2.3.20. *Let $\langle E_1, E_2\rangle$ be a dual system of L^0-modules. Then $\langle \mathscr{U}_{\langle E_1,E_2\rangle}{\Uparrow}\rangle_{\mathcal{A}}$ is a name for the weak topology induced by $\langle E_1{\uparrow}, E_2{\uparrow}\rangle_{\mathcal{A}}$.*

In particular, if $S \subset E_1$ is stable, it is satisfied:

1. *S is $\sigma_{\mathfrak{s}}(E_1, E_2)$-open if and only if* $[\![S{\uparrow}$ is $\sigma(E_1{\uparrow}, E_2{\uparrow})$-open$]\!] = I$;
2. *S is $\sigma_{\mathfrak{s}}(E_1, E_2)$-closed if and only if* $[\![S{\uparrow}$ is $\sigma(E_1{\uparrow}, E_2{\uparrow})$-closed$]\!] = I$;
3. *S is $\sigma_{\mathfrak{s}}(E_1, E_2)$-stably compact if and only if* $[\![S{\uparrow}$ is $\sigma(E_1{\uparrow}, E_2{\uparrow})$-compact$]\!] = I$.

Proof. Take a basic neighborhood of $0 \in E_1$

$$U := U_{(F_k),(a_k),\varepsilon} = \left\{x \in E_1 \colon \sum 1_{a_k} \sup_{y\in F_k} |\langle x, y\rangle| \le \varepsilon\right\},$$

where $(a_k) \in p(I)$, (F_k) is a countable family of non-empty finite subsets of E_2 and $\varepsilon \in L^0_{++}$.

For any $k \in \mathbb{N}$, put $F_k = \{y^k_1, \ldots, y^k_{n_k}\}$. Set $\mathfrak{m} := \sum 1_{a_k} m_k \in L^0(\mathbb{N})$. For any $\mathfrak{n} \in L^0_{\mathfrak{m}}(\mathbb{N})$ define $y_{\mathfrak{n}} := \sum 1_{a_k} \sum_{i=1}^{m_k} 1_{\{l=i\}} y^k_i$. Then the family $(y_{\mathfrak{n}})_{\mathfrak{n}\in L^0_{\mathfrak{m}}(\mathbb{N})}$ is stable and the set $F := \{y_{\mathfrak{n}} \colon \mathfrak{n} \le \mathfrak{m}\}$ is stably finite. A close look allows us to check that

$$U = \left\{x \in E_1 \colon \sup_{y\in F} |\langle x, y\rangle| \le \varepsilon\right\}.$$

By Proposition 2.3.14 $F{\uparrow}$ is a name for a finite set. For any $x \in E_1$, bearing in mind Proposition 2.3.10, one has

$$\left[\!\!\left[\left(\sup_{y\in F} |\langle x, y\rangle|\right)^\bullet = \sup_{y\in F{\uparrow}} |\langle x^\bullet, y\rangle|\right]\!\!\right] = I.$$

Finally, Theorem 2.1.4 allows us to check that

$$\left[\!\!\left[U\!\uparrow = \{x \in E_1\!\uparrow \colon \sup_{y \in F\uparrow} |\langle x, y\rangle| \leq \varepsilon^\bullet\} \right]\!\!\right] = I,$$

in other words, $U\!\uparrow$ is a name for a basic neighborhood for the weak topology $\sigma(E_1\!\uparrow, E_2\!\uparrow)$.

Conversely, if $\mathbb{U}$ is a name for basic neighborhood for the weak topology $\sigma(E_1\!\uparrow, E_2\!\uparrow)$, then there exist names $\mathbb{F}$ and $\mathbb{r}$ for a non-empty finite subset of $E_2\!\downarrow$ and a positive real number, respectively, so that

$$[\![\mathbb{U} = \{x \in E_1\!\uparrow \colon \sup_{y \in \mathbb{F}} |\langle x, y\rangle| \leq \mathbb{r}\}]\!] = I.$$

Set $F := \mathbb{F}\!\downarrow$ and $\varepsilon := \mathbb{r}^\circ$. By Proposition 2.3.14, F is stably finite. Put $F = \{y_{\mathfrak{n}} \colon \mathfrak{n} \in L^0_{\mathfrak{m}}(\mathbb{N})\}$ for some stable sequence $(y_{\mathfrak{n}})_{\mathfrak{n} \in L^0_{\mathfrak{m}}(\mathbb{N})}$. Fixed $k \in \mathbb{N}$, for each $n \leq k$, we define $z^k_n := y_{\mathfrak{m}\wedge n}$. For each $k \in \mathbb{N}$ set $F_k := \{z^k_1, \ldots, z^k_k\}$. Then we have

$$[\![U_{(F_k),(a_k),\varepsilon}\!\uparrow = \mathbb{U}]\!] = I.$$

We conclude that the assignment $U \to U\!\uparrow\downarrow$ provides a bijection between a neighborhood base of $0 \in E_1$ for the stably weak topology and the descent of a name of a neighborhood base of the origin for the weak topology on $E_1\!\uparrow$. The assertion follows.

1,2, and 3 are clear from Proposition 2.3.4. □

Example 2.3.1. *If $(E, \mathscr{T})$ is a locally stable L^0-convex module. Then $\langle E, E^*\rangle$ is dual system of L^0-modules with respect to the bi-L^0-module homomorhism*

$$\langle\cdot,\cdot\rangle : E \times E^* \longrightarrow L^0, \quad (x, \mu) \mapsto \langle x, \mu\rangle := \mu(x).$$

Therefore, we can consider $\langle E\!\uparrow, E^*\!\uparrow\rangle_{\mathcal{A}}$, which is a name for the weak topology on $E\!\uparrow$.

We can argue similarly for the stably weak-$*$ topology.

Comments 2.3.4. *Dual systems of modules have been studied by Kusraev [76]. In particular, [76, Theorem 3.3.10(b)] is related to Theorem 2.2.2 and connects names for dual pairs if real vectors spaces and dual systems of modules.*

Dual systems of L^0-modules with the (ϵ, λ)-topology have been studied by Guo and other co-authors under the name of random dual pairs *(see eg [53, 54]).*

2.4 Instances of application of the Boolean-valued transfer principle

Once we have the 'building blocks', let us see some examples to exhibit how they can be assembled to give rise to different statements of a full development of a module analogue of convex analysis. Of course, this list is not exhaustive and we can create many other pieces of our puzzle. We will review several results of convex analysis and we will see how they are interpreted as theorems on locally stable L^0-convex modules.

We start by a module analogue of the Brouwer's fixed point theorem.

The so-called Schauder–Tychonoff theorem (see eg [19, Theorem 1.12.8]) asserts that:

> "Let (X, τ) be a Hausdorff locally convex space and C a convex compact subset of X, then for any continuous function $\lambda : C \to C$ there exists $y \in C$ such that $\lambda(y) = y$"

We have the following module analogue of the Schauder–Tychonoff theorem:

Theorem 2.4.1. *Let $(E,\mathscr{T})$ be a Hausdorff locally stable L^0-convex module and S an L^0-convex stably compact subset of E, then for any stable continuous function $f:S\to S$ there exists $x\in S$ such that $f(x)=x$.*

Suppose that $(E,\mathscr{T})$ is a Hausdorff locally stable L^0-convex module and $\mathscr{U}$ is a stable neighborhood of $0\in E$ consisting of stable subsets. We have that $\langle\mathscr{U}\Uparrow\rangle_{\mathcal{A}}$ is a name for a Hausdorff locally convex topology (see Propositions 1.4.1 and 2.3.3); $S\uparrow$ is a name for a non-empty subset of $E\uparrow$, moreover, $[\![S\uparrow \text{ is convex}]\!]=I$ and $[\![S\uparrow \text{ is compact}]\!]=I$; finally, $f\uparrow$ is a name for a continuous function from $S\uparrow$ to $S\uparrow$. We see that the result above is just a reformulation of the statement $[\![\text{Schauder–Tychonoff theorem}]\!]=I$. Indeed, by applying the transfer principle first we have that $[\![(\exists x\in S\uparrow)(f\uparrow(x)=x)]\!]=I$. By the maximum principle, we can find $x\in S$ such that $[\![f\uparrow(x^\bullet)=x^\bullet]\!]=I$. Then, we have that $f(x)=x$.

A version of Brouwer Fixed Point Theorem for $(L^0)^d$ was provided in [33], which corresponds to the finite-dimensional case in our context and for which the theorem above is an extension.

Obviously, Theorem 2.4.1 is just an example: we can state a version of any theorem φ on locally convex spaces and it immediately renders a version for locally L^0-modules of the form $[\![\varphi]\!]=I$.

Next, we will show that the main theorems of [36] are also consequences of the transfer principle. For instance, we will see that Theorems 2.8, 3.7 and 3.8 in [36] follow by applying this method.

Recall the classical hyperplane separation theorem (see eg [101, Theorem 3,4]):

> "If C,K are non-empty convex subsets of a locally convex space $(X,\mathscr{T})$ with C closed, K compact, and C and K have empty intersection, then there is a continuous lineal function $\theta:X\to\mathbb{R}$ and positive $r\in\mathbb{R}$ such that $\theta(x)>\theta(y)+r$ for all $x\in C$, $y\in K$".

Then we have the following:

Theorem 2.4.2. *Let $(E,\mathscr{T})$ be a locally stable L^0-convex module, and suppose that S_1,S_2 are stable L^0-convex subsets with S_1 closed and S_2 stably compact. If*

$$1_aS_1\cap 1_aS_2=\emptyset \quad \text{for all } a\in\mathcal{A}\setminus\{0\},$$

then there exists a continuous L^0-module homomorphism $\mu:E\to L^0$ and $\varepsilon\in L^0_{++}$ such that

$$\mu(x)>\mu(y)+\varepsilon \quad \text{for all } x\in S_1, y\in S_2.$$

Note that, if S_1,S_2 are stable subsets of E, then $1_aS_1\cap 1_aS_2=\emptyset$ for all $a\in\mathcal{A}\setminus\{0\}$ if, and only if, $[\![S_1\uparrow\cap S_2\uparrow=\emptyset]\!]=I$. What we have in the statement above is nothing else but a reformulation of the statement $[\![\text{separation theorem}]\!]=I$, which holds due to the transfer principle. The maximum principle provides as with the name $\mathbb{f}$ for a linear function and a name $\mathbb{r}$ for a positive real number such that

$$[\![(\forall x\in S_1\uparrow)(\forall y\in S_2\uparrow)(\mathbb{f}(x)>\mathbb{f}(y)+\mathbb{r})]\!]=I.$$

Then $\mu := \mathbb{f}{\downarrow}$ and $\varepsilon := \mathbb{r}^{\circ}$ satisfy the desired.

This result is a generalization of [36, Theorem 2.8], which proves the case in which S_2 is a singleton.

Recall the conventional version of the Fenchel-Moreau theorem (see eg [9, Theorem 2.22]):

> "If (X, τ) is a locally convex space and $\lambda : X \to \overline{\mathbb{R}}$ is a proper lower semi-continuous convex function, then $\lambda^{**} = \lambda$".

[36, Theorem 3.8] is a modular version of the classical Fenchel-Moreau theorem. We have the following:

Theorem 2.4.3. *If $(E, \mathscr{T})$ is a locally stable L^0-convex module and $f : E \to \bar{L}^0$ is a proper lower semi-continuous and L^0-convex function, then $f^{**} = f$.*

The result above is just a reformulation of the statement $[\![$Fenchel-Moreau theorem$]\!] = I$ and does not need a proof just by considering the name $f{\uparrow}$ and noting that $[\![(f^{**}){\uparrow} = (f{\uparrow})^{**}]\!] = I$ due to Proposition 2.3.18.

The notion of L^0-barrel was introduced in [36]. Namely, a subset S of E is an L^0*-barrel* if it is L^0-convex, L^0-absorbing, L^0-balanced and closed. We will say that a topological L^0-module is *stably L^0-barrelled* if every stable L^0-barrel is a neighborhood of $0 \in E$.

The so-called Fenchel-Rockafellar theorem, see eg [20, Theorem 1] asserts the following:

> "If (X, τ) is a barrelled locally convex space and $\lambda : X \to \overline{\mathbb{R}}$ is proper, lower semi-continuous and convex, then $\partial\lambda(x)$ is non-empty for all $x \in \overset{o}{\mathrm{dom}}(\lambda)$".

The following result, which is a generalization of [36, Theorem 3.7], follows from the transfer principle applied to the statement above:

Theorem 2.4.4. *Let $(E, \mathscr{T})$ be a locally stable L^0-convex module which is stably L^0-barrelled . Let $f : E \to \bar{L}^0$ be a proper lower semi-continuous L^0-convex function. Then,*

$$\partial f(x) \neq \emptyset \quad \text{for all } x \in \overset{o}{\mathrm{dom}}(f).$$

The notion of L^0*-barrelled* topological L^0-module was introduced in [36]; namely, $E[\mathscr{T}]$ is L^0-barrelled if every L^0-barrel is a neighborhood of $0 \in E$. Thus, the notion of stably L^0-barrelled topological L^0-module is more general. This was already pointed out in [56], where the statement above was proven by using the techniques introduced in [36].

Until now we have proven a modular version of the Brouwer fixed point theorem and we have derived the main results of [36] by means of the transfer principle. However, the number of instances of application can be easily increased.

Next, by applying our method, we will provide a list of modular versions of basic results of functional analysis, which hold —are theorems— due to the transfer principle.

Recall the Heine-Borel theorem:

> "Let $(X, |\cdot|)$ be a finite-dimensional normed space, then a subset C of X is compact if and only if C is closed and norm bounded."

The transfer principle yields the following module analogue of the Heine-Borel theorem:

Theorem 2.4.5. *Let $(E, \|\cdot\|)$ be stably finitely generated L^0-normed module. A stable subset S of E is stably compact if and only if S is close and L^0-norm bounded.*

Recall that the Baire Category Theorem asserts:

> "If $(X, |\cdot|)$ is a Banach space and (E_n) is a countable family of closed subsets of X such that $\bigcup_{n\in\mathbb{N}} E_n = X$. Then, there exists a ball $B_r(x)$ and $n \in \mathbb{N}$ such that $B_r(x) \subset E_n$."

This theorem is interpreted in L^0-modules as follows:

Theorem 2.4.6. *Let $(E, \|\cdot\|)$ be an L^0-normed module which is also stably complete. Suppose that $(S_\mathfrak{n})_{\mathfrak{n}\in L^0(\mathbb{N})}$ is a stable family of stable closed subsets of S so that $S = \bigcup_{\mathfrak{n}\in L^0(\mathbb{N})} S_\mathfrak{n}$. Then there exists a ball $B_\varepsilon(x)$ and $\mathfrak{n} \in L^0(\mathbb{N})$ such that $B_\varepsilon(x) \subset S_\mathfrak{n}$.*

The Banach-Alaouglu theorem asserts:

> "If $(X, |\cdot|)$ is a normed space, then $B_{X^*} := \{\theta \in X^* \colon |\theta| \leq 1\}$ is $\sigma(X^*, X)$-compact."

The following theorem is a modular analogue of the Banach-Alaoglu theorem. It is a consequence of the transfer principle and does not need a proof.

Theorem 2.4.7. *Let $(E, \|\cdot\|)$ be an L^0-normed module. Then $B_{E^*} := \{\mu \in E^* \colon \|\mu\| \leq 1\}$ is $\sigma_\mathfrak{s}(E^*, E)$-stably compact.*

Recall the Eberlein-Šmulian theorem:

> "Let $(X, |\cdot|)$ be a normed space and let $C \subset X$ be non-empty. Then C is $\sigma(X, X^*)$-compact if and only if C is sequentially $\sigma(X, X^*)$-compact."

Say that an L^0-normed module $(E, \|\cdot\|)$ is $\sigma_\mathfrak{s}(E, E^*)$-*stably sequentially compact*, if every stable sequence $(x_\mathfrak{n})_{\mathfrak{n}\in L^0(\mathbb{N})}$ admits a convergent stable subsequence $(x_{\mathfrak{n}_\mathfrak{m}})_{\mathfrak{m}\in L^0(\mathbb{N})}$. The theorem above is interpreted in L^0-modules obtaining:

Theorem 2.4.8. *Let $(E, \|\cdot\|)$ be an L^0-normed and $S \subset E$ stable. Then the following are equivalent.*

(i) S is $\sigma_\mathfrak{s}(E, E^*)$-stably compact.

(ii) S is $\sigma_\mathfrak{s}(E, E^*)$-stably sequentially compact.

Recall the classical James' compactness theorem:

> "Let $(X, |\cdot|)$ be a Banach space and $C \subset X$ be non-empty and $\sigma(X, X^*)$-closed. Then C is $\sigma(X, X^*)$-compact if and only if for any $\theta \in X^*$ there exists $x_0 \in C$ such that $\theta(x_0) = \sup_{x\in C} \theta(x)$".

The James' compactness theorem is interpreted as follows:

Theorem 2.4.9. *Let $(E, \|\cdot\|)$ be a stably complete L^0-normed module and $S \subset E$ be stable and $\sigma_\mathfrak{s}(E, E^*)$-closed. Then S is $\sigma_\mathfrak{s}(E, E^*)$-stably compact if and only if for any $\mu \in E^*$ there exists $x_0 \in S$ such that $\mu(x_0) = \sup_{x\in S} \mu(x)$.*

Some perturbed versions of James's compactness theorem have been provided in literature, see eg [68, Theorem A.1], [86, Theorem 2] and [87]. It most general version (see [103, Theorem 2.4] and [84]) asserts the following:

> "Let $(X, |\cdot|)$ be a Banach space and $\lambda : X \to \overline{\mathbb{R}}$ a proper function. If for every $\theta \in X^*$ there is an $x_0 \in X$ such that $\theta(x_0) - \lambda(x_0) = \lambda^*(\theta)$, then λ the set $V_r(\lambda) := \{x \in X : \lambda(x) \leq r\}$ is relatively $\sigma(X, X^*)$-compact for all $r \in \mathbb{R}$".

The theorem above gives rise to the following:

Theorem 2.4.10. *Let $(E, \|\cdot\|)$ be a stably complete L^0-normed module and let $f : E \to \bar{L}^0$ be a proper function with the local property. If for every $\mu \in E^*$ there is an $x_0 \in E$ such that $\mu(x_0) - f(x_0) = f^*(\mu)$, then the set $V_\eta(f) = \{x \in E : f(x) \leq \eta\}$ is relatively $\sigma_{\mathfrak{s}}(E, E^*)$-stably compact for all $\eta \in L^0$ with $V_\eta(f) \neq \emptyset$.*

The following statement is an interpretation of the known fact that every lower semi-continuous attains a minimum on a compact set:

Theorem 2.4.11. *Let $(E, \mathscr{T})$ be a locally stable L^0-convex topology, S a stable subset of E, and $f : S \to L^0$ a stable function. If S is stably compact and f is lower semi-continuous, then there exists $x_0 \in S$ so that $f(x_0) = \inf\{f(x) : x \in S\}$.*

A version for $(L^0)^d$ was obtained in [23], which is a particular case of the statement above for the finite dimensional case.

Recall the Krein-Šmulian theorem:

> "Let $(X, |\cdot|)$ be a Banach space and $C \subset X^*$ be non-empty and convex. Then, the following are equivalent:
>
> 1. C is $\sigma(X^*, X)$-closed;
> 2. $C \cap \{\theta \in X^* : |\theta| \leq r\}$ is $\sigma(X^*, X)$-closed for all positive real number r."

The Krein-Šmulian theorem has the following interpretation:

Theorem 2.4.12. *Let $(E, \|\cdot\|)$ be a stably complete L^0-normed module and $S \subset E^*$ be stable and L^0-convex. Then the following conditions are equivalent:*

1. *S is $\sigma_{\mathfrak{s}}(E^*, E)$-closed;*
2. *$S \cap \{\mu \in E^* : |\mu| \leq \varepsilon\}$ is $\sigma_{\mathfrak{s}}(E^*, E)$-closed for all $\varepsilon \in L^0_{++}$.*

The classical Mazur theorem assets:

> "Let (X, τ) be a locally convex space and C a convex subset of X, then $\overline{C}^{\tau} = \overline{C}^{\sigma(X, X^*)}$."

In Chapter 1 we established a module analogue of the Mazur Theorem for L^0-normed modules with additive class of stable subsets (see Corollary 1.3.1). If we replace the condition of having additive class of stable subsets by the stronger condition of having the countable concatenation property, we obtain the following Mazur theorem:

Theorem 2.4.13. *Let $(E, \mathscr{T})$ be a locally stable L^0-convex module and S a stable L^0-convex subset of E. Then $\overline{S}^{\mathscr{T}} = \overline{S}^{\sigma_{\mathfrak{s}}(E,E^*)}$.*

Recall the well-known functional analysis result for dual pairs:

> "Let $\langle X, Y\rangle$ be a dual pair, then $\theta \in X^*[\sigma(X, Y)]$ if and only if there exists $y \in Y$ such that $\theta(x) = \langle x, y\rangle$ for all $x \in X$."

We have the following:

Theorem 2.4.14. *Let $\langle E_1, E_2\rangle$ be a dual system of L^0-modules, then $\mu \in E_1^*[\sigma_{\mathfrak{s}}(E_1, E_2)]$ if and only if there exists $y \in E_2$ such that $\mu(x) = \langle x, y\rangle$ for all $x \in E_1$.*

Chapter 3

A Boolean-valued model approach to Conditional set theory

Conditional set theory was introduced in [32]. In this framework, an abstraction of the notion of countable concatenation property is introduced and all the structures are supposed to be stable under concatenations with respect to an underlying complete Boolean algebra $\mathcal{A}$. Conditional set theory gives meaning to conditional versions of different classical mathematical objects (conditional sets, conditional functions, conditional binary relations, conditional topological spaces, and so on), and uses a formulation which consistently depends on the elements of $\mathcal{A}$. This allows us to formulate conditional analogues of classical results, which are proven by adapting existing classical proofs of them.

In this chapter, we study the precise connection between conditional sets and Boolean-valued models and show that conditional set theory can be seen as a particular instance of Boolean-valued models. This amounts to a Boolean-valued transfer principle from classical set theory to conditional set theory: the conditional version of any result proven by conventional means is also true.

In Section 3.1, we recall the notions of conditional set and conditional function. We show that any conditional set $\mathbf{X}$ can be made into a non-empty set $\mathbf{X}\uparrow$ within $V^{(\mathcal{A})}$ and any conditional function $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ can be made into a function $\mathbf{f}\uparrow$ within $V^{(\mathcal{A})}$. We show that the ascent functor $\cdot\uparrow$ is in fact an equivalence between the category of conditional sets and conditional functions and the category of non-empty sets and functions within $\overline{V}^{(\mathcal{A})}$.

In Section 3.2, we establish a Boolean-valued transfer principle from classical set theory to conditional set theory. Namely, we review the different elements of conditional set theory and construct suitable ascent operations for all of them. This gives a precise interpretation within $V^{(\mathcal{A})}$ of any conditional version of classical mathematical objects: conditional binary relations are interpreted as binary relations, conditional topologies are interpreted as topologies and so on, in the model $V^{(\mathcal{A})}$. In other words, we show that conditional set theory is a Boolean-valued interpretation of classical set theory. All this shows that the conditional version of any classical theorem is again a theorem due to the transfer principle of $V^{(\mathcal{A})}$.

In Section 3.3, we show some instances of application of the Boolean-valued transfer principle. We provide conditional versions of the Brouwer's fixed point theorem, the open map theorem, the Baire category theorem, the Uniform Boundedness principle, the Eberlein-Šmulian theorem, the Amir-Lindenstrauss theorem and of a perturbed version of James' compactness theorem. All of them hold without the necessity of a step-by-step proof.

Finally, in Section 3.4 we give a precise connection between locally stable L^0-convex modules and conditional locally convex spaces. Namely, we find that, for a fixed probability algebra, there is an equivalence between the category of locally stable L^0-modules and continuous L^0-module homomorphisms and the category of conditionally locally convex spaces and conditionally linear continuous functions.

3.1 Conditional sets, Boolean metric spaces and names for non-empty sets

Throughout this section we will consider a complete Boolean algebra $(\mathcal{A}, \vee, \wedge, {}^c, I, 0)$ (for a review of Boolean algebras, see Section A.6). Recall that the ideal of $\mathcal{A}$ generated by $a \in \mathcal{A}$ is denoted by $\mathcal{A}_a$.

An elementary fact that we will use several times is the following:

Proposition 3.1.1. *Let $(a_i)_{i\in\Delta}$ be a family of elements of $\mathcal{A}$ and set $a = \bigvee_{i\in\Delta} a_i$. Then there exists $(b_i)_{i\in\Delta} \in p(a)$ such that $b_i \leq a_i$ for all $i \in \Delta$.*

Proof. By the well-ordering theorem (see Section A.3) we can endow Δ with a well-ordering. Then, for each $i \in \Delta$, we define $b_i := a_i \wedge (\bigvee_{j<i} b_j)^c$. □

Let us recall the notion of conditional set, which was introduced in [32].

Definition 3.1.1. *Let X be a non-empty set. A* conditional set *of X (and $\mathcal{A}$) is a set $\boldsymbol{X}$ such that there exists a surjective mapping*

$$X \times \mathcal{A} \longrightarrow \boldsymbol{X} \quad (x, a) \mapsto x|a$$

satisfying the following:

(C1) if $x, y \in X$ and $a, b \in \mathcal{A}$ with $x|a = y|b$, then $a = b$;

(C2) (Consistency) if $x, y \in X$ and $a, b \in \mathcal{A}$ with $a \leq b$, then $x|b = y|b$ implies $x|a = y|a$;

(C3) (Stability) if for any partition $(a_i) \in p(I)$ and family (x_i) of elements of X, there exists a unique $x \in X$ such that $x|a_i = x_i|a_i$ for all i.

The unique element $x \in X$ provided by (C3) is called the concatenation of (x_i) along (a_i) and is denoted by $\sum x_i|a_i$. An element of the form $x|I$ is called conditional element *of $\boldsymbol{X}$ and is denoted by $\boldsymbol{x}$.*

Remark 3.1.1. *Suppose that $\boldsymbol{X}$ is a conditional set of a non-empty set X. Then the set of all conditional elements of $\boldsymbol{X}$ is in bijection with X, that is, the map $x \mapsto \boldsymbol{x}$ is a bijection between both sets. Indeed, clearly is surjective. To prove the injectivity, suppose that $x|I = y|I$, then the uniqueness part of (C3) yields that $x = y$.*

Definition 3.1.2. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be conditional sets.*

- *A function $f : X \to Y$ is said to be* stable, *if*

$$f\left(\sum x_k|a_k\right) = \sum f(x_k)|a_k \quad \text{for all } (a_k) \in p(I) \text{ and } (x_k) \subset X.$$

- *If* $f : X \to Y$ *is stable, then one has that* $\boldsymbol{G_f} := \{(x|a, f(x)|a)\colon x \in X,\ a \in \mathcal{A}\}$ *is a conditional set of the graph* G_f *of* f*. Then* $\boldsymbol{G_f}$ *is called the* conditional function *generated by* f *and is denoted by* $\boldsymbol{f}\colon \boldsymbol{X} \to \boldsymbol{Y}$*.*

- $\boldsymbol{f}$ *is* conditionally injective *if* f *is injective;* $\boldsymbol{f}$ *is* conditionally surjective *if* f *is surjective; and* $\boldsymbol{f}$ *is* conditionally bijective *if* f *is bijective.*

- *If* $\boldsymbol{f}\colon \boldsymbol{X} \to \boldsymbol{Y}$ *is a conditional function,* $\boldsymbol{S} \sqsubset \boldsymbol{X}$ *and* $\boldsymbol{L} \sqsubset \boldsymbol{Y}$ *we define:*

  $\boldsymbol{f}(\boldsymbol{S})$ *the conditional set generated by the stable subset* $f(S)$*.*

  $\boldsymbol{f}^{-1}(\boldsymbol{L})$ *the conditional set generated by the stable subset* $f^{-1}(L)$*.*

Definition 3.1.3. *Let* $\boldsymbol{X}$ *be a conditional set. For any two* $x, y \in X$*, we define:*

$$a_{x,y} := \bigvee\{a \in \mathcal{A}\colon x|a = y|a\}.$$

Proposition 3.1.2. *Let* $\boldsymbol{X}$ *be a conditional set. For any two* $x, y \in X$*, it is satisfied that*

$$x|a = y|a \quad \text{whenever } a \leq a_{x,y}. \tag{3.1}$$

Moreover, for all $x, y, z \in X$ *the following is satisfied:*

1. $a_{x,y} = a_{y,x}$*;*

2. $a_{x,y} = I$ *if and only if* $x = y$*;*

3. $a_{x,z} \geq a_{x,y} \wedge a_{y,z}$*.*

Proof. Let $x, y \in X$. Take $(a_i) \in p(a_{x,y})$ with $x|a_i = y|a_i$, each i. For each i, take $b_i := a \wedge a_i$. Then $(b_i) \in p(a)$. By (C2), we have that $x|b_i = y|b_i$, each i. By (C3) we can set $z := x|a + y|a^c$. Since $z|a = x|a$, by (C2), we have that $z|b_i = x|b_i = y|b_i$ for each i. Then, by the uniqueness part of (C3), we have that $z = y$, and by (C2), one has $y|a = z|a = x|a$.

1 is clear. To prove 2, suppose that $a_{x,y} = I$. Due to (3.1), $x|I = y|I$. By the uniqueness part of (C3), we have that $x = y$.

Finally, prove 3. Given x, y, z, due to (3.1) one has that $x|a_{x,y} = y|a_{x,y}$ and $y|a_{y,z} = z|a_{y,z}$. Set $a := a_{x,y} \wedge a_{y,z}$. Due to (C2), one has that $x|a = y|a = z|a$. We conclude that $a \leq a_{x,z}$. □

Notice that the conditions 1-3 of Proposition 3.1.2 somehow resemble the definition of a metric. A well-known notion is the following:

Definition 3.1.4. *Let* X *be a non-empty set. A function* $d : X \times X \to \mathcal{A}$ *is called a* Boolean metric *if for every* $x, y, z \in X$ *the following hold:*

(BM1) $d(x, y) = d(y, x)$*;*

(BM2) $d(x, y) = 0$ *if and only if* $x = y$*;*

(BM3) $d(x, z) \leq d(x, y) \vee d(y, z)$*.*

The pair (X, d) *is said to be a* Boolean metric space*. We say that a Boolean metric space* (X, d) *has the* concatenation property *if for any* $(a_i) \in p(I)$ *and* $(x_i) \subset X$*, there exists* $x \in X$ *such that* $d(x_i, x) \wedge a_i = 0$ *for all* i*.*

Remark 3.1.2. *Let (X, d) be a Boolean metric space with the concatenation property. Suppose that $(a_i) \in p(I)$ and $(x_i) \subset X$, and take a concatenation $x \in X$ of (x_i) with $d(x, x_i) \wedge a_i = 0$ for all i. Notice that in the definition above we do not require x to be unique. This is justified due to the fact that, as proven in [81, 3.4.2], x is necessarily unique. Indeed, if $y \in X$ also satisfies that $d(y, x_i) \wedge a_i = 0$ for all i, then, by (BM3), it follows*

$$a_i \wedge d(x, y) \leq a_i \wedge (d(x, x_i) \vee d(y, x_i)) = 0, \quad \text{each } i,$$

and thus $d(x, y) = 0$, hence $x = y$.

Next, we will show that the structures of conditional set and Boolean metric space with the concatenation property are essentially the same.

Proposition 3.1.3. *Let $\boldsymbol{X}$ be a conditional set of a non-empty set X. Define*

$$d_X : X \times X \longrightarrow \mathcal{A} \quad d_X(x, y) := a_{x,y}^c.$$

Then (X, d_X) is a Boolean metric space with the concatenation property.

Proof. (BM1), (BM2) and (BM3) are direct consequences of Proposition 3.1.2.

To prove the concatenation property, take $(a_i) \in p(I)$ and $(x_i) \subset X$. Set $x := \sum x_i|a_i$. Clearly, $a_i \leq a_{x,x_i}$ for all i, hence $d_X(x, x_i) \wedge a_i = 0$ for all $i \in I$. □

Proposition 3.1.4. *Let (X, d) be a Boolean metric space with the concatenation property. Define the following equivalence relation on $X \times \mathcal{A}$:*

$$(x, a) \sim (y, b) \quad \text{whenever } a = b,\ d(x, y) \wedge a = 0.$$

Denote by $x|a$ the equivalence class of (x, a). Then the quotient set $\boldsymbol{X}$ is a conditional set.

Proof. (C1) follows by definition. Suppose that $x|a = y|a$ and $b \leq a$, then $d(x, y) \wedge b \leq d(x, y) \wedge a = 0$. Thus $x|b = y|b$ and (C2) follows.

Finally, to prove (C3) take $(a_i) \in p(I)$ and $(x_i) \subset X$. Then, by the concatenation property, there exists a $x \in X$ such that $a_i \wedge d(x, x_i) = 0$ for all i. This means that $x|a_i = x_i|a_i$ for all i. If $y \in X$ also satisfies $x|a_i = x_i|a_i$ for all i, it follows that $a_i \wedge d(x, x_i) = 0$ for all i. By Remark 3.1.2, we have that $x = y$. □

Example 3.1.1. *Suppose that $\mathcal{A}$ is the probability algebra associated to a probability space. If E is an L^0-module with the countable concatenation property, then*

$$d_E : E \times E \longrightarrow \mathcal{F}, \quad (x, y) \mapsto \bigvee\{a \in \mathcal{A} \colon 1_a x = 1_a y\}$$

is a Boolean metric, and the countable concatenation property of E amounts to the concatenation property of (E, d_E). We can therefore define the corresponding conditional set $\boldsymbol{E}$.

Definition 3.1.5. *Let (X, d_X) and (Y, d_Y) be Boolean metric spaces. A function $f : X \to Y$ is said to be* contractive *if*

$$d_Y(f(x), f(y)) \leq d_X(x, y) \quad \text{for all } x, y \in X.$$

Proposition 3.1.5. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be conditional sets of non-empty sets X and Y, respectively, and $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ a conditional function. Then $f : X \to Y$ is a contractive function w.r.t.the corresponding Boolean metrics d_X and d_Y.*

Proof. Take $x, y \in X$ and prove that $d_Y(f(x), f(y)) \leq d_X(x,y)$. It suffices to show that $a_{x,y} \leq a_{f(x),f(y)}$. Since f is stable one has

$$\begin{aligned} f(x)|a_{x,y} &= f(x|a_{x,y} + x|a_{x,y}^c)|a_{x,y} \\ &= f(y|a_{x,y} + x|a_{x,y}^c)|a_{x,y} \\ &= (f(y)|a_{x,y} + f(x)|a_{x,y}^c)|a_{x,y} = f(y)|a_{x,y}, \end{aligned}$$

where we have used above that $x|a_{x,y} = y|a_{x,y}$. This implies that $a_{x,y} \leq a_{f(x),f(y)}$, as desired. $\square$

Proposition 3.1.6. *Let (X, d_X) and (Y, d_Y) be Boolean metric spaces with the concatenation property and $f : X \to Y$ a contractive function. Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be the corresponding conditional sets. Then f is stable and it therefore defines a conditional function $\boldsymbol{f}\colon \boldsymbol{X} \to \boldsymbol{Y}$.*

Proof. Take $(x_i) \subset X$ and $(a_i) \in p(I)$, and prove that $f(\sum x_i|a_i) = \sum f(x_i)|a_i$. Set $x := \sum x_i|a_i$. We know that $d_X(x, x_i) \wedge a_i$ for each i. Then, since f is contractive, one has

$$d_Y(f(x_i), f(x)) \wedge a_i \leq d_X(x_i, y_i) \wedge a_i = 0, \text{ each } i.$$

We conclude that $f(x_i)|a_i = f(x)|a_i$ for all i, which implies the assertions. $\square$

Denote by **Cond**, CBMet and $\overline{V}_*^{(\mathcal{A})}$ the category of conditional sets and conditional functions, the category of Boolean metric spaces with the concatenation property and contractive functions, and the category of names for non-empty sets and names for functions of $\overline{V}^{(\mathcal{A})}$, respectively.

On one hand, a close look shows that Propositions 3.1.3, 3.1.4, 3.1.5 and 3.1.6 establish an equivalence between the categories **Cond** and CBMet. On the other, it is known that CBMet and $\overline{V}_*^{(\mathcal{A})}$ are equivalent categories (see [81, Theorem 3.5.10]). Therefore, we conclude that **Cond** and $\overline{V}_*^{(\mathcal{A})}$ are also equivalent categories.

Since we are interested in the underlying construction of the equivalence between **Cond** and $\overline{V}_*^{(\mathcal{A})}$, we will explicitly show next the precise form of this equivalence, which will enable a Boolean transfer principle from classical set theory to conditional set theory.

In the following, we define the *ascent of a conditional set.*

Let $\mathbf{X}$ be a conditional set, we define the following:

- For any $\mathbf{x} \in \mathbf{X}$, let $\mathbf{x}^\bullet$ denote the unique element in $\overline{V}^{(\mathcal{A})}$ equivalent to the name

$$\{\check{y} \colon y \in X\} \longrightarrow \mathcal{A} \quad \check{y} \mapsto a_{x,y}.$$

- We define the ascent of $\mathbf{X}$, denoted by $\mathbf{X}\!\uparrow$, to be the unique member of $\overline{V}^{(\mathcal{A})}$ equivalent to the name

$$\{\mathbf{x}^\bullet \colon x \in X\} \longrightarrow \mathcal{A} \quad \mathbf{x}^\bullet \mapsto I.$$

Lemma 3.1.1. *Let $\boldsymbol{X}$ be a conditional set. Then*

$$[\![\boldsymbol{x}^\bullet = \boldsymbol{y}^\bullet]\!] = a_{x,y} \quad \textit{for all } x, y \in X.$$

Proof.

$$[\![\mathbf{x}^\bullet = \mathbf{y}^\bullet]\!] = \bigwedge_{u\in X} (a_{x,u} \Rightarrow [\![\check{u} \in \mathbf{y}^\bullet]\!]) \wedge \bigwedge_{v\in X} (a_{y,v} \Rightarrow [\![\check{v} \in \mathbf{x}^\bullet]\!])\,. \tag{3.2}$$

In addition,

$$[\![\check{u} \in \mathbf{y}^\bullet]\!] = \bigvee_{w\in X} a_{y,w} \wedge [\![\check{u} = \check{w}]\!] = a_{y,u},$$

because $[\![\check{u} = \check{w}]\!] = I$ if $u = w$, $[\![\check{u} = \check{w}]\!] = 0$ otherwise. Similarly, one has $[\![\check{v} \in \mathbf{x}^\bullet]\!] = a_{x,v}$.

Therefore, replacing in (3.2), one has

$$[\![\mathbf{x}^\bullet = \mathbf{y}^\bullet]\!] = \bigwedge_{u\in X} (a^c_{x,u} \vee a_{y,u}) \wedge \bigwedge_{v\in X} (a^c_{y,v} \vee a_{x,v}).$$

By considering above $u = x$ and $v = y$, we obtain

$$[\![\mathbf{x}^\bullet = \mathbf{y}^\bullet]\!] \leq a_{x,y}.$$

For the reverse inequality, suppose by contradiction that $0 < a := a_{x,y} \wedge (a^c_{x,u} \vee a_{y,u})^c$ for some u. Since $a \leq a_{x,y}, a_{x,u}$ one has $y|a = x|a = u|a$ by Proposition 3.1.2. But $a \leq a^c_{y,u}$ implies that $y|a \neq u|a$, which is a contradiction. □

Proposition 3.1.7. *Let $\boldsymbol{X}$ be a conditional set. Then the mapping $i_X : X \to \boldsymbol{X}\uparrow\downarrow$, $x \mapsto \boldsymbol{x}^\bullet$, is a bijection. Moreover, for any $(a_i) \in p(I)$ and $(x_i) \subset X$ one has*

$$i_X\left(\sum x_i|a_i\right) = \sum i_X(x_i)a_i. \tag{3.3}$$

Proof. If $\mathbf{x}^\bullet = \mathbf{y}^\bullet$, then, by the previous lemma, on has $a_{x,y} = [\![\mathbf{x}^\bullet = \mathbf{y}^\bullet]\!] = I$. This implies that $\mathbf{x} = \mathbf{y}$.

To prove the surjectivity, take $\mathbb{z} \in \mathbf{X}\uparrow\downarrow$. Then

$$I = [\![\mathbb{z} \in \mathbf{X}\uparrow]\!] = \bigvee_{x\in X} [\![\mathbf{x}^\bullet = \mathbb{z}]\!].$$

We can take $(a_i) \in p(I)$ and $(x_k) \subset X$ such that $a_i \leq [\![\mathbf{x}^\bullet = \mathbb{z}]\!]$ for all i. Take $x = \sum x_i|a_i$. One has $[\![\mathbf{x}^\bullet = \mathbf{x}_i^\bullet]\!] = a_{x,x_i} \geq a_i$, each i. Due to the mixing principle, we finally obtain that $\mathbb{z} = \mathbf{x}^\bullet$.

An identical argument proves (3.3). □

Proposition 3.1.8. *If $\mathbb{x}$ is a name for a non-empty set, then there exists a conditional set* $\mathbf{cond}(\mathbb{x}\downarrow)$ *of $\mathbb{x}\downarrow$ such that $\sum \mathbb{u}_i a_i = \sum \mathbb{u}_i|a_i$ for all $(a_i) \in p(I)$ and $(\mathbb{u}_i) \subset \mathbb{x}\downarrow$.*

Proof. Consider the equivalent relation on $\mathbb{x}\downarrow \times \mathcal{A}$ defined by

$$(\mathbb{u}, a) = (\mathbb{v}, b) \quad \text{whenever } a = b,\ [\![\mathbb{u} = \mathbb{v}]\!] \geq a.$$

Let $\mathbf{cond}(\mathbb{x}\downarrow)$ be the quotient set and consider the corresponding projection $(\mathbb{u}, a) \mapsto \mathbb{u}|a$. Then $\mathbf{cond}(\mathbb{x}\downarrow)$ is a conditional set. Indeed, (C1) and (C2) follow easily from the definition. To prove (C3), take $(a_i) \in p(I)$ and $(\mathbb{u}_i) \subset \mathbb{x}$. The mixing principle provides us with $\mathbb{u} := \sum \mathbb{u}_i a_i$. Then, using Lemma 3.1.1, it is simple to verify that $\mathbb{u}|a_i = \mathbb{u}_i|a_i$ for all i. □

The following proposition is a direct consequence of Propositions 3.1.7 and 3.1.8.

Corollary 3.1.1. *Let $\boldsymbol{X}$ be a conditional set. The function $i_X : X \to \boldsymbol{X}\uparrow\downarrow$ is a stable bijection. Therefore it defines a conditional bijection $\boldsymbol{i_X} : \boldsymbol{X} \to$* $\mathbf{cond}(\boldsymbol{X}\uparrow\downarrow)$.

Corollary 3.1.2. *Let $\mathbb{x}$ be a name for a non-empty set. The function*

$$i_{\mathbb{x}\downarrow} : \mathbb{x}\downarrow \to \mathbf{cond}(\mathbb{x}\downarrow)\uparrow\downarrow \quad \mathbb{u} \to (\mathbb{u}|I)^{\bullet}$$

is an extensional bijection, thus $\mathbb{i}_{\mathbb{x}} := i_{\mathbb{x}\downarrow}\uparrow$ is a name for a bijection from $\mathbb{x}$ to $\mathbf{cond}(\mathbb{x}\downarrow)\uparrow$ such that $[\![\mathbb{i}_{\mathbb{x}}\uparrow(\mathbb{u}) = \mathbb{i}_{\mathbb{x}}(\mathbb{u})^{\bullet}]\!] = I$ for all $\mathbb{u} \in \mathbb{x}\downarrow$.

Proposition 3.1.9. *Let $\boldsymbol{f}\colon \boldsymbol{X} \to \boldsymbol{Y}$ be a conditional function. Then there exists a name $\boldsymbol{f}\uparrow$ for a function from $\boldsymbol{X}\uparrow$ to $\boldsymbol{Y}\uparrow$ such that*

$$[\![\boldsymbol{f}\uparrow(\boldsymbol{x}^{\bullet}) = \boldsymbol{f}(\boldsymbol{x})^{\bullet}]\!] = I \quad \text{for all } \boldsymbol{x} \in \boldsymbol{X}.$$

Proof. The mapping $\mathbf{x}^{\bullet} \mapsto \mathbf{f}(\mathbf{x})^{\bullet}$ is an extensional function from $\mathbf{X}\uparrow\downarrow$ to $\mathbf{Y}\uparrow\downarrow$. Theorem 2.1.5 provides us with the desired name $\mathbf{f}\uparrow$. □

Proposition 3.1.10. *Let $\mathbb{X}$, $\mathbb{Y}$ be names for non-empty sets and $\mathbb{f}$ a name for a function from $\mathbb{X}$ to $\mathbb{Y}$. Then there exists a stable function $\mathbb{f}\downarrow : \mathbb{X}\downarrow \to \mathbb{Y}\downarrow$ such that*

$$[\![\mathbb{f}\downarrow(\mathbb{x}) = \mathbb{f}(\mathbb{x})]\!] \quad \text{for all } \mathbb{x} \in \mathbb{X}\downarrow.$$

Theorem 3.1.1. *For fixed a complete Boolean algebra $\mathcal{A}$. The functor $\Phi(\boldsymbol{X}) = \boldsymbol{X}\uparrow$, $\Phi(\boldsymbol{f}) = \boldsymbol{f}\uparrow$ is an equivalence of categories between* **Cond** *and $\overline{V}^{(\mathcal{A})}_{*}$.*

Proof. The functor Φ is well-defined by construction. Consider the functor $\Psi(\mathbb{x}) = \mathbf{cond}(\mathbb{x}\downarrow)$, $\Psi(\mathbb{f}) = \mathbf{cond}(\mathbb{f}\downarrow)$. Propositions 3.1.8 and 3.1.10 show that Ψ is a functor from $\overline{V}^{(\mathcal{A})}_{*}$ to **Cond**.

We have that $\Psi\Phi$ is naturally isomorphic to the identity functor of **Cond**. Indeed, for fixed a conditional set $\mathbf{X}$, we consider $\mathbf{i_X} : \mathbf{X} \to \mathbf{cond}(\mathbf{X}\uparrow\downarrow)$, which is a conditional bijection due to Corollary 3.1.1. Inspection shows that the family of conditional bijections $\mathbf{i_X}$ is a natural transformation of the identity.

Now, given a name $\mathbb{x}$ for a non-empty set, we consider $\mathbb{i}_{\mathbb{x}}$, which is a name for a bijection from $\mathbb{x}$ to $\mathbf{cond}(\mathbb{x}\downarrow)\uparrow$ due to Corollary 3.1.2. Inspection shows that the family of names for bijections $\mathbb{i}_{\mathbb{x}}$ is a natural transformation.

□

Comments 3.1.1. *The concept of a Boolean metric appeared at the beginning of the 1950s. It was introduced with the purpose of studying various distances given on abstract sets and taking values in partially ordered sets [18, 34, 93, 7]. Boolean metric spaces and their relation with Boolean-valued models are thoroughly studied in [81]. In particular, they provide an useful tool for defining different types of ascent operations for several mathematical structures.*

3.2 A Boolean-valued transfer principle from classical set theory to conditional set theory

The important message of Theorem 3.1.1 is not the equivalence of categories provided, but that for any conditional set $\mathbf{X}$ one can construct a tailored name $\mathbf{X}\uparrow$ for a non-empty set such that $\mathbf{X}$ and $\mathbf{cond}(\mathbf{X}\uparrow\downarrow)$ are essentially the same conditional set.

This will allow one to interpret the different conditional versions of different mathematical objects as names for classical mathematical objects within $V^{(\mathcal{A})}$.

On this basis, in the following we will review many conditional versions of classical notions introduced in [32], and we will construct different ascent operations which will give us a suitable

Boolean-valued interpretation. Of course, this list is not exhaustive and one can define more and more ascents for new elements of conditional set theory.

Throughout we will use the same symbol $\uparrow$ for denoting many ascent operations of the same provenance but with different formal definitions. Therefore, if $\mathbf{X}$ is a conditional set, the writing $\mathbf{X}\uparrow$ will be correctly understood only if one knows which type of conditional set $\mathbf{X}$ is.

3.2.1 Conditional subsets and conditional collection of conditional subsets

Let $\mathbf{X}$ be a conditional set of a non-empty set X.

- A non-empty subset Y of X is said to be *stable* if

$$\sum x_i|a_i \in Y \quad \text{whenever } (a_i)\in p(I),\ (x_i)\subset Y.$$

- If Y is a stable subset of X, then the set

$$\mathbf{Y} := \{y|a\colon a\in\mathcal{A},\ y\in Y\}$$

  is a conditional set of Y which is called a *conditional subset* of $\mathbf{Y}$. We write $\mathbf{Y}\sqsubset\mathbf{X}$.

- For any conditional subset $\mathbf{Y}$ of $\mathbf{X}$ and $a\in\mathcal{A}$, let

$$\mathbf{Y}|a := \{x|b\colon x\in Y,\ b\in\mathcal{A}_a\}\,.$$

  Denote by $\mathfrak{S}(\mathbf{X})$ the collection of all stable subsets of X. Then

$$\boldsymbol{\mathcal{P}}(\mathbf{X}) = \{\mathbf{Y}|a\colon Y\in\mathfrak{S}(\mathbf{X}),\ a\in\mathcal{A}\}\,,$$

  is a conditional set of $\mathfrak{S}(\mathbf{X})$ which is called the *conditional power set* of $\mathbf{X}$.

- A conditional subset $\boldsymbol{\mathcal{C}}$ of $\boldsymbol{\mathcal{P}}(\mathbf{X})$ is called a *conditional collection of conditional subsets* of $\mathbf{X}$.

Notice that if $\mathbf{Y}$ is a conditional subset and $a\in\mathcal{A}$, then $\mathbf{Y}|a$ is a conditional set of $Y|a := \{y|a\colon y\in Y\}$ with respect to the complete Boolean algebra $\mathcal{A}_a$. In this case, we say that $\mathbf{Y}|a$ is a conditional subset of $\mathbf{X}$ on a.

It is known that $\sqsubset$ is a partial order on $\boldsymbol{\mathcal{P}}(\mathbf{X})$. Note, that this follows from the fact that the relation $\sqsubset$ is precisely the classical inclusion "$\subset$" between members of $\boldsymbol{\mathcal{P}}(\mathbf{X})$.

Recall the conditional set operations:

Definition 3.2.1. *Let $\boldsymbol{X}$ be a conditional set. Given a non-empty family $(\boldsymbol{Y}_i|a_i)$ of elements of $\boldsymbol{\mathcal{P}}(\boldsymbol{X})$ we define its:*

- Conditional union *to be* $\bigsqcup \boldsymbol{Y}_i|a_i := \boldsymbol{Y}|a$ *where* $a:=\bigvee_i a_i$ *and, for fixed* $z\in X$*,*

$$Y := \left\{\sum y_i|b_i + z|a^c\colon (b_i)\in p(a) \text{ with } b_i\le a_i,\ y_i\in Y_i \text{ for each } i\right\}.$$

- Conditional intersection *to be* $\bigsqcap \boldsymbol{Y}_i|a_i := \boldsymbol{Y}|a$ *where*

$$a := \bigvee\{b \colon b \leq \bigwedge a_i,\ \textit{there exists } x \in X \textit{ such that } x|b \in \boldsymbol{Y}_i|b \textit{ for all } i\}$$

and

$$Y := \{x \in X \colon x|a \in \boldsymbol{Y}_i|a \textit{ for all } i\}.$$

Given a conditional subset $\boldsymbol{Y}|a$ *of* $\boldsymbol{X}$ *we define its* conditional complement *to be*

$$(\boldsymbol{Y}|a)^{\sqsubset} := \bigsqcup\{\boldsymbol{Z}|c \in \boldsymbol{\mathcal{P}}(\boldsymbol{X}) \colon \boldsymbol{Z}|c \sqcap \boldsymbol{Y}|a = \boldsymbol{X}|0\}$$

It is known that $(\boldsymbol{\mathcal{P}}(\mathbf{X}), \sqcup, \sqcap, \mathbf{X}, \mathbf{X}|0)$ has structure of complete Boolean algebra.

Next, we will see how the notions above are interpreted within the Boolean-valued universe $V^{(\mathcal{A})}$.

For a given conditional set $\mathbf{X}$, we will define ascent operations for conditional subsets of $\mathbf{X}$ and conditional collections of conditional subsets of $\mathbf{X}$:

- *Ascent of a conditional subset*: Suppose that $\mathbf{Y}$ is a conditional subset of $\mathbf{X}$, then we denote by $\mathbf{Y}\uparrow$ the unique member of $\overline{V}^{(\mathcal{A})}$ equivalent to

$$\{\mathbf{x}^\bullet \colon x \in Y\} \longrightarrow \mathcal{A} \quad \mathbf{x}^\bullet \mapsto I.$$

- *Ascent of a conditional collection of conditional subsets*: Suppose that $\boldsymbol{\mathcal{C}}$ is a conditional collection of conditional subsets of $\mathbf{X}$, we denote by $\boldsymbol{\mathcal{C}}\Uparrow$ to be the unique member of $\overline{V}^{(\mathcal{A})}$ equivalent to

$$\{\mathbf{Y}\uparrow \colon Y \in \mathcal{C}\} \longrightarrow \mathcal{A} \quad \mathbf{Y}\uparrow \mapsto I.$$

The following results show how the ascents above are interpreted within the Boolean-valued universe. The proofs of them are omitted as they are elementary by using the basic principles of Boolean-valued models such as we did in Chapter 2.

Proposition 3.2.1. *Let* $\boldsymbol{X}$ *be a conditional set. The following properties hold:*

1. *If* $\boldsymbol{Y}$ *is a conditional subset of* $\boldsymbol{X}$*, then* $\boldsymbol{Y}\uparrow$ *is a name for a non-empty subset of* $\boldsymbol{X}\uparrow$.
2. *If* $\boldsymbol{\mathcal{C}}$ *is conditional collection of conditional subsets of* $\boldsymbol{X}$*, then* $\boldsymbol{\mathcal{C}}\Uparrow$ *is a name for a non-empty collection of non-empty subsets of* $\boldsymbol{X}\uparrow$.

   In particular, $\boldsymbol{\mathcal{P}}(\boldsymbol{X})\Uparrow$ *is a name for the set of all non-empty subsets of* $\boldsymbol{X}\uparrow$. *That is* $[\![\mathcal{P}(\boldsymbol{X}\uparrow) \setminus \{\emptyset\} = \boldsymbol{\mathcal{P}}(\boldsymbol{X})\Uparrow]\!] = I$.

Moreover, $\cdot\uparrow$ *is a bijection between conditional subsets of* $\boldsymbol{X}$ *and names for non-empty subsets of* $\boldsymbol{X}\uparrow$*, and* $\cdot\Uparrow$ *is a bijection between conditional collections of conditional subsets of* $\boldsymbol{X}$ *and non-empty collections of non-empty subsets of* $\boldsymbol{X}\uparrow$.

Proposition 3.2.2. *Let* $\boldsymbol{X}$ *be a conditional set. The following properties hold:*

1. *If* $\boldsymbol{Y}_1$*,* $\boldsymbol{Y}_2$ *are conditional subsets of* $\boldsymbol{X}$*, then*

$$[\![(\boldsymbol{Y}_1 \sqcup \boldsymbol{Y}_2)\uparrow = \boldsymbol{Y}_1\uparrow \cup \boldsymbol{Y}_2\uparrow]\!] = I.$$

Moreover, $\boldsymbol{Y}_1 \sqcap \boldsymbol{Y}_2$ *is on* I *if and only if* $[\![\boldsymbol{Y}_1\uparrow \cap \boldsymbol{Y}_2\uparrow \neq \emptyset]\!] = I$*. In this case*

$$[\![(\boldsymbol{Y}_1 \sqcap \boldsymbol{Y}_2)\uparrow = \boldsymbol{Y}_1\uparrow \cap \boldsymbol{Y}_2\uparrow]\!] = I,$$

In addition, $\boldsymbol{Y}_1 \sqcap \boldsymbol{Y}_2$ *is on* 0 *if and only if* $[\![\boldsymbol{Y}_1\uparrow \cap \boldsymbol{Y}_2\uparrow = \emptyset]\!] = I$.

2. If $\boldsymbol{\mathcal{C}}$ is a conditional collection of conditional subsets of $\boldsymbol{X}$, then

$$\left[\!\!\left[(\bigsqcup_{\boldsymbol{O}\in\boldsymbol{\mathcal{C}}} \boldsymbol{O})\!\uparrow = \bigcup_{O\in\boldsymbol{\mathcal{C}}\Uparrow} O \right]\!\!\right] = I.$$

Moreover, $\bigsqcap_{\boldsymbol{O}\in\boldsymbol{\mathcal{C}}} \boldsymbol{O}$ is on I if and only if $\left[\!\!\left[\bigcap_{O\in\boldsymbol{\mathcal{C}}\Uparrow} O \neq \emptyset\right]\!\!\right] = I$. In this case

$$\left[\!\!\left[(\bigsqcap_{\boldsymbol{O}\in\boldsymbol{\mathcal{C}}} \boldsymbol{O})\!\uparrow = \bigcap_{O\in\boldsymbol{\mathcal{C}}\Uparrow} O \right]\!\!\right] = I.$$

In addition, $\bigsqcap_{\boldsymbol{O}\in\boldsymbol{\mathcal{C}}} \boldsymbol{O}$ is on 0 if and only if $\left[\!\!\left[\bigcap_{O\in\boldsymbol{\mathcal{C}}\Uparrow} O = \emptyset\right]\!\!\right] = I$.

3. If $\boldsymbol{Y}$ is a conditional subset of $\boldsymbol{X}$ and $\boldsymbol{Y}^{\sqsubset}$ is on I, then $[\![\boldsymbol{Y}^{\sqsubset}\!\uparrow = (\boldsymbol{Y}\!\uparrow)^c]\!] = I$.

Remark 3.2.1. *In the proposition above we have only considered the cases where $\boldsymbol{Y}_1 \sqcap \boldsymbol{Y}_2$ is either on I or 0. Obviously, it can be also on $1 < a < \mathcal{A}$. Whereas this case is very useful in the constructive approach to conditional set theory such as in [32], this case is uninteresting for the Boolean-valued transfer principle because in set theory intersections are only either empty or non-empty. Thus the transcription of any ZFC theorem into the conditional setting will only involve conditional intersections either on I or 0.*

3.2.2 Conditional set of step functions, conditional natural numbers and conditional rational numbers

In the following we recall the notion of *conditional set of steps functions* of a non-empty set X.

Given a non-empty set X we consider the set of all families (x_i, a_i) of elements in $X \times \mathcal{A}$ where $(a_i) \in p(I)$. On this collection we define the equivalence relation

$$(x_i, a_i)_i \sim (y_j, b_j)_j \quad \text{whenever } x_i = y_j \text{ for all } i, j \text{ with } a_i \wedge b_j > 0.$$

Then the quotient set, denoted by $L^0_{\mathfrak{s}}(X)$, is called the *set of steps functions* on S. Let $[x_i, a_i]$ denote the class of (x_i, a_i). We can make $L^0_{\mathfrak{s}}(X)$ into a conditional set $\mathbf{L}^0_{\mathfrak{s}}(X)$ by defining

$$[x_i, a_i]|a := \{[y_j, b_j] \colon x_i = y_j \text{ for all } i, j \text{ with } a \wedge a_i \wedge b_j > 0\}.$$

Notice that each element of $L^0_{\mathfrak{s}}(X)$ can be written as $\sum [x_i, I]|a_i$. Thus, the elements of $L^0_{\mathfrak{s}}(X)$ are denoted by $\sum x_i|a_i$.

In particular, if X is either $\mathbb{N}$ or $\mathbb{Q}$, we obtain the corresponding conditional sets $\mathbf{N} := \mathbf{L}^0_{\mathfrak{s}}(\mathbb{N})$ and $\mathbf{Q} := \mathbf{L}^0_{\mathfrak{s}}(\mathbb{Q})$ called *conditional natural numbers* and *conditional rational numbers*, respectively.

Although $\mathbf{L}^0_{\mathfrak{s}}(X)$ is a conditional set and we can therefore consider its ascent, next we will define a specific ascent operation for this type of conditional set, which will provide a more practical interpretation in the Boolen-valued universe.

Namely, we define the ascent of $\mathbf{L}^0_{\mathfrak{s}}(X)$ simply by $\mathbf{L}^0_{\mathfrak{s}}(X)\!\uparrow := \check{X}$. Recall that $\check{\cdot}$ denotes the canonical immersion of V into $\overline{V}^{(\mathcal{A})}$.

The following justifies this definition:

Proposition 3.2.3. *Given a non-empty set X the mapping*

$$L^0_{\mathfrak{s}}(X) \longrightarrow \check{X}\downarrow \quad \sum x_i|a_i \mapsto \sum \check{x}_i a_i$$

is well defined and is a stable bijection.

Proof. One has

$$\begin{aligned}
\left[\!\!\left[\sum \check{x}_i a_i = \sum \check{y}_j b_j\right]\!\!\right] = I \quad & \text{if and only if} \quad a_i, b_j \leq [\![\check{x}_i = \check{y}_j]\!] \text{ for all } i,j \text{ with } a_i \wedge b_j > 0 \\
& \text{if and only if} \quad I = [\![\check{x}_i = \check{y}_j]\!] \text{ for all } i,j \text{ with } a_i \wedge b_j > 0 \\
& \text{if and only if} \quad x_i = y_j \text{ for all } i,j \text{ with } a_i \wedge b_j > 0 \\
& \text{if and only if} \quad \sum x_i|a_i = \sum y_j|b_j.
\end{aligned}$$

This proves that the map is well-defined and injective.
The surjectivity is a consequence of the mixing principle.
It follows by inspection that the mapping is also stable. □

Recall that $[\![\check{\mathbb{N}} = \mathbb{N}]\!] = [\![\check{\mathbb{Q}} = \mathbb{Q}]\!] = I$. Consequently, as an immediate consequence of the proposition above, we have:

Corollary 3.2.1. *The following functions are well-define and are stable bijections*

$$L^0_{\mathfrak{s}}(\mathbb{N}) \longrightarrow \mathbb{N}_{\mathcal{A}}\downarrow \quad \sum n_i|a_i \mapsto \sum \check{n}_i a_i$$

$$L^0_{\mathfrak{s}}(\mathbb{Q}) \longrightarrow \mathbb{Q}_{\mathcal{A}}\downarrow \quad \sum q_i|a_i \mapsto \sum \check{q}_i a_i$$

The corollary above justifies to define $\mathbf{N}\uparrow := \mathbb{N}_{\mathcal{A}}$ and $\mathbf{Q}\uparrow := \mathbb{Q}_{\mathcal{A}}$.

3.2.3 Conditional binary relations

Given two conditional sets $\mathbf{X}$ and $\mathbf{Y}$, we define its *conditional Cartesian product* to be

$$\mathbf{X} \times \mathbf{Y} := \{(x|a, y|a) \colon a \in \mathcal{A},\ x \in X,\ y \in Y\}.$$

Clearly, $\mathbf{X} \times \mathbf{Y}$ is a conditional set of $X \times Y$.

Definition 3.2.2. *A* conditional binary relation *on $\boldsymbol{X}$ is a conditional subset $\boldsymbol{T}$ of $\boldsymbol{X} \times \boldsymbol{X}$. Given $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$ and $a \in \mathcal{A}$, we say that $\boldsymbol{x}$ and $\boldsymbol{y}$ are in* conditional relation *on a if $(x|a, y|a) \in \boldsymbol{T}$, in which case we write $\boldsymbol{x}|a\boldsymbol{T}\boldsymbol{y}|a$. A conditional binary relation $\boldsymbol{T}$ on $\boldsymbol{X}$ is* conditionally reflexive*,* conditionally symmetric*,* conditionally antisymmetric*, or* conditionally transitive *if T is reflexive, symmetric, antisymmetric, or transitive, respectively, on X.*

Definition 3.2.3. *A conditional binary relation $\leq$ on $\boldsymbol{X}$ is said to be a* conditional direction *if $\leq$ is a direction on X. In this case, we say that $\boldsymbol{X}$ is* conditionally directed*.*

A conditional binary relation $\leq$ on $\boldsymbol{X}$ is said to be a conditional partial order *if $\leq$ is a partial order on X. In this case we say that $\boldsymbol{X}$ is* conditionally partially ordered*.*

Given a conditional partial order $\leq$ on $\boldsymbol{X}$, we write $\boldsymbol{x}|a < \boldsymbol{y}|a$ if $\boldsymbol{x}|a \leq \boldsymbol{y}|a$ and $a_{x,y} \wedge a = 0$. A conditional partial order $\leq$ is conditionally total *if for any two conditional elements $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$ there exists $(a,b) \in p(a^c_{x,y})$ such that $\boldsymbol{x}|a < \boldsymbol{y}|a$ and $\boldsymbol{y}|b < \boldsymbol{x}|b$. In this case, we say that $\boldsymbol{X}$ is* conditionally totally ordered*.*

Definition 3.2.4. *Let $\leq$ be a conditionally partial order on $\boldsymbol{X}$ and $\boldsymbol{Y}$ a conditional subset of $\boldsymbol{X}$. A conditional element $\boldsymbol{z} \in \boldsymbol{X}$ is a:*

1. conditional upper bound *(resp. conditional lower bound) of $\boldsymbol{Y}$ if z is a upper (resp. lower) bound of Y with respect to the partial order $\leq$;*
2. conditional supremum *(resp. conditional infimum) of $\boldsymbol{Y}$ if z is a supremum (resp. infimum) of Y with respect to the partial order $\leq$.*
3. *$(\boldsymbol{X}, \leq)$ is* conditionally Dedekind complete *if $(X, \leq)$ is Dedekind complete.*

Next, we will see how the notions above are interpreted within the Boolean-valued universe $V^{(\mathcal{A})}$.

Ascent of a conditional binary relation: If $\mathbf{T}$ is a conditional binary relation on $\mathbf{X}$ we define $\mathbf{T}\!\uparrow$ to be the unique member of $\overline{V}^{(\mathcal{A})}$ equivalent to the name

$$\{(\mathbf{x}^\bullet, \mathbf{y}^\bullet)_{\mathcal{A}} \colon (\mathbf{x}, \mathbf{y}) \in \mathbf{T}\} \to \mathcal{A} \quad (\mathbf{x}^\bullet, \mathbf{y}^\bullet)_{\mathcal{A}} \mapsto I.$$

The following proposition justifies the definition above.

Proposition 3.2.4. *Let $\boldsymbol{T}$ be a conditional relation on $\boldsymbol{X}$. The mapping*

$$\boldsymbol{T} \longrightarrow \boldsymbol{T}\!\uparrow\downarrow \quad (\boldsymbol{x}, \boldsymbol{y}) \mapsto (\boldsymbol{x}^\bullet, \boldsymbol{y}^\bullet)_{\mathcal{A}}$$

is a stable bijection.

Proof. By Proposition 2.1.3 the map is injective an stable; by Remark 2.1.2 it is stable. By means of the mixing principle one proves that it is surjective. □

The following results can be proven by means of an elementary manipulation of Boolean truth values. Thus, we omit the proofs.

Proposition 3.2.5. *Let $\boldsymbol{T}$ be a conditional binary relation on $\boldsymbol{X}$. Then $\boldsymbol{T}\!\uparrow$ is a name for a binary relation on $\boldsymbol{X}\!\uparrow$. Moreover:*

1. *$\boldsymbol{T}$ is conditionally reflexive if and only if* $[\![\boldsymbol{T}\!\uparrow \text{ is reflexive}]\!] = I$;
2. *$\boldsymbol{T}$ is conditionally symmetric if and only if* $[\![\boldsymbol{T}\!\uparrow \text{ is symmetric}]\!] = I$;
3. *$\boldsymbol{T}$ is conditionally antisymmetric if and only if* $[\![\boldsymbol{T}\!\uparrow \text{ is antisymmetric}]\!] = I$;
4. *$\boldsymbol{T}$ is conditionally transitive if and only if* $[\![\boldsymbol{T}\!\uparrow \text{ is transitive}]\!] = I$;
5. *$\boldsymbol{T}$ is a conditional direction if and only if* $[\![\boldsymbol{T}\!\uparrow \text{ is a direction}]\!] = I$;
6. *$\boldsymbol{T}$ is conditional partial order if and only if* $[\![\boldsymbol{T}\!\uparrow \text{ is a partial order}]\!] = I$.

Proposition 3.2.6. *Let $\leq$ be a conditional partial order on $\boldsymbol{X}$ and $\boldsymbol{Y}$ a conditional subset of $\boldsymbol{X}$. Then:*

1. *$\leq$ is conditionally total if and only* $[\![\leq\uparrow \text{ is total}]\!] = I$;
2. *$\boldsymbol{z} \in \boldsymbol{X}$ is a conditional upper bound of $\boldsymbol{Y}$ if and only if* $[\![\boldsymbol{z}^\bullet \text{ is an upper bound of } \boldsymbol{Y}\!\uparrow]\!] = I$;
3. *$\boldsymbol{z} \in \boldsymbol{X}$ is a conditional lower bound of $\boldsymbol{Y}$ if and only if* $[\![\boldsymbol{z}^\bullet \text{ is a lower bound of } \boldsymbol{Y}\!\uparrow]\!] = I$;
4. *$\boldsymbol{z} \in \boldsymbol{X}$ is a conditional supremum of $\boldsymbol{Y}$ if and only if* $[\![\boldsymbol{z}^\bullet \text{ is a supremum of } \boldsymbol{Y}\!\uparrow]\!] = I$;
5. *$\boldsymbol{z} \in \boldsymbol{X}$ is a conditional infimum of $\boldsymbol{Y}$ if and only if* $[\![\boldsymbol{z}^\bullet \text{ is a infimum of } \boldsymbol{Y}\!\uparrow]\!] = I$;
6. *$(\boldsymbol{X}, \leq)$ is conditionally Dedekind complete if and only if* $[\![(\boldsymbol{X}\!\uparrow, \leq\uparrow)_{\mathcal{A}} \text{ is Dedekind complete}]\!] = I$.

3.2.4 Conditional families

Conditional functions are also called *conditional families*. Let $\mathbf{X}$ and $\mathbf{J}$ be conditional sets. If the mapping $j \mapsto x_j$ from J to X is stable, then the family $(x_j)_{j\in J}$ is called a *stable family*. The corresponding conditional family is denoted by $(\mathbf{x_j})_{\mathbf{j}\in\mathbf{J}}$, or simply $(\mathbf{x_j})$.

Suppose that $(\mathbf{x_j})_{\mathbf{j}\in\mathbf{J}}$ is a conditional family of conditional elements of $\mathbf{X}$ and $a \in \mathcal{A}$. We can consider $\mathbf{J}|a$ and $\mathbf{X}|a$, which are conditional sets with respect to the complete Boolean algebra $\mathcal{A}_a$. Then

$$J|a \longrightarrow X|a \quad j|a \mapsto x_j|a$$

is a stable function, and it therefore generates a conditional family $(\mathbf{x_j})_{\mathbf{j}\in\mathbf{J}}|a$.

Define

$$\mathbf{fun}(\mathbf{J},\mathbf{X}) := \{(\mathbf{x_j})_{\mathbf{j}\in\mathbf{J}}|a\colon (\mathbf{x_j})_{\mathbf{j}\in\mathbf{J}} \text{ is a conditional family and } a \in \mathcal{A}\}.$$

Then $\mathbf{fun}(\mathbf{J},\mathbf{X})$ is a conditional set of the set of stable families $(x_j)_{j\in J}$.

Let $\mathbf{X}$ be a conditional set and $(\mathbf{Y_j})_{\mathbf{j}\in\mathbf{J}}$ a conditional family of conditional subsets of $\mathbf{X}$. Then we define the *conditional Cartesian product*

$$\prod_{\mathbf{j}\in\mathbf{J}} \mathbf{Y_j} := \{(\mathbf{x_j})|a\colon (\mathbf{x_j}) \in \mathbf{fun}(\mathbf{J},\mathbf{X}),\ \mathbf{x_j} \in \mathbf{Y_j} \text{ for all } \mathbf{j} \in \mathbf{J},\ a \in \mathcal{A}\}\,.$$

Now, let us define suitable ascents:

- Given a conditional subset $\boldsymbol{\mathcal{H}}$ of $\mathbf{fun}(\mathbf{J},\mathbf{X})$ we define its ascent, say $\boldsymbol{\mathcal{H}}\!\uparrow$, to be the unique member of $\overline{V}^{(\mathcal{A})}$ equivalent to

$$\{\mathbf{f}\!\uparrow\colon \mathbf{f} \in \boldsymbol{\mathcal{H}}\} \longrightarrow \mathcal{A} \quad \mathbf{f}\!\uparrow \mapsto I.$$

- $\prod_{\mathbf{j}\in\mathbf{J}} \mathbf{Y_j}$ is a conditional subset of $\mathbf{fun}(\mathbf{J},\mathbf{X})$, thus one can consider $\left(\prod_{\mathbf{j}\in\mathbf{J}} \mathbf{Y_j}\right)\uparrow$.

The following result can be proven by using an elementary manipulation of Boolean truth values; we omit the proof.

Proposition 3.2.7. *Let $\boldsymbol{X}$ and $\boldsymbol{J}$ be conditional sets, and $(\boldsymbol{Y_j})_{\boldsymbol{j}\in\boldsymbol{J}}$. Then*

1. $[\![\mathbf{fun}(\boldsymbol{J},\boldsymbol{X})\!\uparrow = \mathrm{fun}(\boldsymbol{J}\!\uparrow,\boldsymbol{X}\!\uparrow)]\!] = I$;
2. $\boldsymbol{\mathcal{H}}\!\uparrow$ *is a name for a non-empty subset of* $\mathrm{fun}(\boldsymbol{J}\!\uparrow,\boldsymbol{X}\!\uparrow)$;
3. $\left(\prod_{\boldsymbol{j}\in\boldsymbol{J}} \boldsymbol{Y_j}\right)\uparrow$ *is a name for the Cartesian product of the family* $(\boldsymbol{Y_j})\!\uparrow$.

The transfer principle applied to the Axiom of Choice proves the conditional axiom of choice provided in [32].

Proposition 3.2.8. *[32, Theorem 2.26] Let $\boldsymbol{X}$ be a conditional set and $(\boldsymbol{Y_j})_{\boldsymbol{j}\in\boldsymbol{J}}$ a conditional family of conditional subsets of $\boldsymbol{X}$, then there exists a conditional family $(\boldsymbol{x_j})$ of conditional elements of $\boldsymbol{X}$ such that $\boldsymbol{x_j} \in \boldsymbol{Y_j}$ for all $\boldsymbol{j} \in \boldsymbol{J}$.*

Comments 3.2.1. *In [32], the conditional Cartesian product is defined differently. Namely, given a conventional family of $(\boldsymbol{X}_j)_{j\in J}$ of conditional sets its conditional Cartesian product is defined by*

$$\prod_{j\in J} \boldsymbol{Y}_j := \{(x_i|a)\colon x_i \in X_j,\ a \in \mathcal{A}\}.$$

However, this definition is not compatible with the conditional axiom of choice. This is the reason we adopt a different definition.

3.2.5 Conditional cardinality

Definition 3.2.5. *Two conditional sets $\boldsymbol{X}$, $\boldsymbol{Y}$ have the same conditional cardinality, if there a conditional bijection between both.*

Given a conditional natural number $\mathbf{m} \in \mathbf{N}$, we define the conditional denote by $\mathbf{N_m}$ the conditional subset of $\mathbf{N}$ generated by the stable subset

$$N_m := \{n \in N \colon n \leq m\}.$$

Definition 3.2.6. *A conditional set $\boldsymbol{X}$ is said to be:*

- conditionally countable *if there is a conditional surjection $\boldsymbol{n} \mapsto \boldsymbol{x_n}$ from $\boldsymbol{N}$ to $\boldsymbol{X}$;*
- conditionally finite *if there exists a conditional surjection $\boldsymbol{n} \mapsto \boldsymbol{x_n}$ from $\boldsymbol{N_m}$ to $\boldsymbol{X}$ for some conditional natural number $\boldsymbol{m} \in \boldsymbol{N}$.*

Proposition 3.2.9. *Let $\boldsymbol{X}$, $\boldsymbol{Y}$ be conditional sets:*

1. *$\boldsymbol{X}$ and $\boldsymbol{Y}$ have the same conditional cardinality if and only if* $[\![\boldsymbol{X}\!\uparrow$ and $\boldsymbol{Y}\!\uparrow$ have the same cardinality$]\!] = I$;
2. *$\boldsymbol{X}$ is conditionally countable if and only if* $[\![\boldsymbol{X}\!\uparrow$ is countable$]\!] = I$;
3. *$\boldsymbol{X}$ is conditionally finite if and only if* $[\![\boldsymbol{X}\!\uparrow$ is finite$]\!] = I$.

Proof. In each case, it suffices to consider the ascent of the corresponding conditional function. □

3.2.6 Conditional topologies

Definition 3.2.7. *Let $\boldsymbol{X}$ be a conditional set:*

- *A conditional collection $\boldsymbol{\mathcal{B}}$ of conditional subsets of $\boldsymbol{X}$ is called a* conditional topological base *on $\boldsymbol{X}$ if:*
  1. *For every $\boldsymbol{x} \in \boldsymbol{X}$ there exists $\boldsymbol{U} \in \boldsymbol{\mathcal{B}}$ such that $\boldsymbol{x} \in \boldsymbol{U}$;*
  2. *If $\boldsymbol{U}, \boldsymbol{V} \in \boldsymbol{\mathcal{B}}$ and $\boldsymbol{x} \in \boldsymbol{U} \sqcap \boldsymbol{V}$, then there exists $\boldsymbol{W} \in \boldsymbol{\mathcal{B}}$ such that $\boldsymbol{x} \in \boldsymbol{W}$ and $\boldsymbol{W} \sqsubset \boldsymbol{U} \sqcap \boldsymbol{V}$.*
- *A conditional collection $\boldsymbol{\mathcal{T}}$ of conditional subsets of $\boldsymbol{X}$ is called a* conditional topology *if there exists a conditional topological base $\boldsymbol{\mathcal{B}}$ on $\boldsymbol{X}$ such that*

  $$\boldsymbol{\mathcal{T}} := \{\boldsymbol{O}|a \colon a \in \mathcal{A},\ \boldsymbol{O} \sqsubset \boldsymbol{X}, \text{ for all } \boldsymbol{x} \in \boldsymbol{O} \text{ there exists } \boldsymbol{U} \in \boldsymbol{\mathcal{B}} \text{ such that } \boldsymbol{x} \in \boldsymbol{U} \sqsubset \boldsymbol{O}\}.$$

  In this case we say that $\boldsymbol{\mathcal{B}}$ is a conditional base *of $\boldsymbol{\mathcal{T}}$ and write $\boldsymbol{\mathcal{T}} = \langle \boldsymbol{\mathcal{B}} \rangle$. The pair $(\boldsymbol{X}, \boldsymbol{\mathcal{T}})$ is called a* conditional topological space.
- *If $(\boldsymbol{X}, \boldsymbol{\mathcal{T}})$ is a conditional topological space, the conditional elements of $\boldsymbol{\mathcal{T}}$ are called* conditionally open sets. *A conditional subset $\boldsymbol{C}$ of $\boldsymbol{X}$ such that $\boldsymbol{C}^{\sqsubset} \in \boldsymbol{\mathcal{T}}$ is called a* conditional closed set.[1]

Definition 3.2.8. *Let $(\boldsymbol{X}, \boldsymbol{\mathcal{T}})$ be a conditional topological space:*

[1]Notice that $\mathbf{C}^{\sqsubset}$ could be on a with $0 < a < I$.

- *A conditional subset* $\boldsymbol{U}$ *of* $\boldsymbol{X}$ *is a* conditional neighborhood *of* $\boldsymbol{x}\in\boldsymbol{X}$ *is there exists* $\boldsymbol{O}\in\boldsymbol{\mathcal{T}}$ *such that* $\boldsymbol{x}\in\boldsymbol{O}\sqsubset\boldsymbol{U}$*. Given* $\boldsymbol{x}\in\boldsymbol{X}$*, we define* $\boldsymbol{\mathcal{U}_x}$ *to be the conditional collection of all conditional neighborhoods of* $\boldsymbol{x}$*.*

- *Given* $\boldsymbol{x}\in\boldsymbol{X}$*, a conditional collection* $\boldsymbol{\mathcal{V}}$ *of conditional subsets of* $\boldsymbol{X}$ *is a* conditional neighborhood base *of* $\boldsymbol{x}$ *if* $\boldsymbol{\mathcal{V}}\sqsubset\boldsymbol{\mathcal{U}_x}$ *and for every* $\boldsymbol{U}\in\boldsymbol{\mathcal{U}_x}$ *there exists* $\boldsymbol{V}\in\boldsymbol{\mathcal{V}}$ *so that* $\boldsymbol{V}\sqsubset\boldsymbol{U}$*.*

- *The conditional topology* $\boldsymbol{\mathcal{T}}$ *is said to be* conditionally Hausdorff *if for every two conditional elements* $\boldsymbol{x},\boldsymbol{y}\in\boldsymbol{X}$ *such that* $a_{x,y}=0$*, there exists* $\boldsymbol{U}\in\boldsymbol{\mathcal{U}_x}$ *and* $\boldsymbol{V}\in\boldsymbol{\mathcal{U}_y}$ *such that* $\boldsymbol{U}\sqcap\boldsymbol{V}=\boldsymbol{X}|0$*.*

Definition 3.2.9. *Let* $(\boldsymbol{X},\boldsymbol{\mathcal{T}})$ *be a conditional topological space and* $\boldsymbol{Y}$ *a conditional subset of* $\boldsymbol{X}$*.*

- *The* conditional interior *of* $\boldsymbol{Y}$ *is defined by*

$$\overset{o}{\boldsymbol{Y}}=\sqcup\{\boldsymbol{O}\colon \boldsymbol{O} \text{ conditionally open}, \boldsymbol{O}\sqsubset\boldsymbol{Y}\}.$$

- *The* conditional closure *of* $\boldsymbol{Y}$ *is defined by*

$$\overline{\boldsymbol{Y}}:=\sqcap\{\boldsymbol{C}\colon \boldsymbol{C} \text{ conditionally closed}, \boldsymbol{Y}\sqsubset\boldsymbol{C}\}.$$

Definition 3.2.10. *Let* $(\boldsymbol{X},\boldsymbol{\mathcal{T}})$ *be a conditional topological space.*

1. *A conditional subset* $\boldsymbol{Y}$ *of* $\boldsymbol{X}$ *is* conditionally dense *if* $\overline{\boldsymbol{Y}}=\boldsymbol{X}$*;*
2. $\boldsymbol{X}$ *is conditionally separable if there exists a conditionally countable dense subset* $\boldsymbol{Y}$ *of* $\boldsymbol{X}$*.*

Next, we will study how the notions above can be interpreted within $V^{(\mathcal{A})}$. The following result can be proven by means of a simple manipulation of Boolean truth values; we omit the proof.

Proposition 3.2.10. *Let* $(\boldsymbol{X},\boldsymbol{\mathcal{T}})$ *be a conditional topological space. Then there exists a name* T *for a topology on* $\boldsymbol{X}\!\uparrow$ *such that* $[\![\boldsymbol{\mathcal{T}}\Uparrow\cup\{\emptyset\}=\mathrm{T}]\!]=I$*. In addition,* $\boldsymbol{\mathcal{T}}$ *is conditionally Hausdorff if and only if* $[\![\mathrm{T}\text{ is Hausdorff}]\!]=I$*. Furthermore, if* $\boldsymbol{Y}$ *is a conditional subset:*

- $\boldsymbol{Y}$ *is conditionally open w.r.t.* $\boldsymbol{\mathcal{T}}$ *if and only if* $[\![\boldsymbol{Y}\!\uparrow\text{ is open w.r.t. }\mathrm{T}]\!]=I$*;*
- $\boldsymbol{Y}$ *is conditionally closed w.r.t.* $\boldsymbol{\mathcal{T}}$ *if and only if* $[\![\boldsymbol{Y}\!\uparrow\text{ is closed w.r.t. }\mathrm{T}]\!]=I$*;*
- $[\![\overline{\boldsymbol{Y}}\!\uparrow=\overline{\boldsymbol{Y}\!\uparrow}]\!]=I$ *and* $\left[\!\!\left[\overset{o}{\boldsymbol{Y}}\!\uparrow=\overset{o}{\widehat{\boldsymbol{Y}\!\uparrow}}\right]\!\!\right]=I$*;*
- $\boldsymbol{Y}$ *is conditionally dense w.r.t.* $\boldsymbol{\mathcal{T}}$ *if and only if* $[\![\boldsymbol{Y}\!\uparrow\text{ is dense w.r.t. }\mathrm{T}]\!]=I$*. In particular,* $\boldsymbol{X}$ *is conditionally separable if and only if* $[\![\boldsymbol{X}\!\uparrow\text{ separable}]\!]=I$*.*

3.2.7 Conditional continuity

Definition 3.2.11. *Let* $(\boldsymbol{X},\boldsymbol{\mathcal{T}_X})$*,* $(\boldsymbol{Y},\boldsymbol{\mathcal{T}_Y})$ *be conditional topological spaces. A conditional function* $\boldsymbol{f}\colon\boldsymbol{X}\to\boldsymbol{Y}$ *is said to be*

- conditionally continuous at $\boldsymbol{x}$ *if for every* $\boldsymbol{V}\in\boldsymbol{\mathcal{U}_{f(x)}}$ *there exists* $\boldsymbol{U}\in\boldsymbol{\mathcal{U}_x}$ *such that* $\boldsymbol{f}(\boldsymbol{U})\sqsubset\boldsymbol{V}$*;*

- conditionally continuous *if* $\boldsymbol{f}$ *is conditionally continuous at any* $\boldsymbol{x} \in \boldsymbol{X}$*.*

The next proposition is proven by using an elementary manipulation of Boolean truth values; we omit the proof.

Proposition 3.2.11. *Let* $(\boldsymbol{X}, \boldsymbol{\mathcal{T}}_{\boldsymbol{X}})$*,* $(\boldsymbol{Y}, \boldsymbol{\mathcal{T}}_{\boldsymbol{Y}})$ *be conditional topological spaces and* $\boldsymbol{f}\colon \boldsymbol{X} \to \boldsymbol{Y}$ *a conditional function:*

1. $\boldsymbol{f}$ *is conditionally continuous at* $\boldsymbol{x} \in \boldsymbol{X}$ *if and only if* $[\![\boldsymbol{f}\!\uparrow \text{ is continuous at } \boldsymbol{x}^\bullet]\!] = I$*;*
2. $\boldsymbol{f}$ *is conditionally continuous if and only if* $[\![\boldsymbol{f}\!\uparrow \text{ is continuous}]\!] = I$*.*

3.2.8 Conditional filters and conditional nets

Definition 3.2.12. *Let* $\boldsymbol{X}$ *be a conditional set:*

- *A* conditional net *in* $\boldsymbol{X}$ *is a conditional family* $(\boldsymbol{x_j})_{\boldsymbol{j} \in \boldsymbol{J}}$ *such that* $\boldsymbol{J}$ *is a conditionally directed set.*
- *A conditional collection* $\boldsymbol{\mathcal{G}}$ *of conditional subsets of* $\boldsymbol{X}$ *is called a* conditional filter base *if for every* $\boldsymbol{U}, \boldsymbol{V} \in \boldsymbol{\mathcal{G}}$ *there exists* $\boldsymbol{W} \in \boldsymbol{\mathcal{G}}$ *such that* $\boldsymbol{W} \sqsubset \boldsymbol{U} \sqcap \boldsymbol{V}$*.*
- *A* conditional sequence *in a conditional set* $\boldsymbol{X}$ *is a conditional net* $(\boldsymbol{x_n})_{\boldsymbol{n} \in \boldsymbol{N}}$ *in* $\boldsymbol{X}$*.*
- *Given two conditional sequences* $(\boldsymbol{x_n})$ *and* $(\boldsymbol{y_n})$ *in* $\boldsymbol{X}$*, we say that* $(\boldsymbol{y_n})$ *is a conditional subsequence of* $(\boldsymbol{x_n})$ *if there exists a conditional sequence* $(\boldsymbol{n_k})$ *in* $\boldsymbol{N}$*, with* $\boldsymbol{n_k} < \boldsymbol{n_{k'}}$ *whenever* $\boldsymbol{k} < \boldsymbol{k}'$*, such that* $\boldsymbol{y_k} = \boldsymbol{x_{n_k}}$ *for all* $\boldsymbol{k} \in \boldsymbol{N}$*.*
- *A conditional collection* $\boldsymbol{\mathcal{F}}$ *of conditional subsets of* $\boldsymbol{X}$ *is called a* conditional filter *if it satisfied the following:*
  1. *If* $\boldsymbol{Y} \in \boldsymbol{\mathcal{T}}$ *and* $\boldsymbol{Y} \sqsubset \boldsymbol{Z} \sqsubset \boldsymbol{X}$*, then* $\boldsymbol{Z} \in \boldsymbol{\mathcal{F}}$*;*
  2. *if* $\boldsymbol{Y}, \boldsymbol{Z} \in \boldsymbol{\mathcal{F}}$*, then* $\boldsymbol{Y} \sqcap \boldsymbol{Z}$ *is on* I *and* $\boldsymbol{Y} \sqcap \boldsymbol{Z} \in \boldsymbol{\mathcal{F}}$*.*
- *A* conditional ultrafilter *on* $\boldsymbol{X}$ *is a conditional filter that is maximal element in the set of all conditional filters on* $\boldsymbol{X}$ *(w.r.t. the partial order* $\sqsubset$*).*

The proofs of the following results follow by a simple manipulation of Boolean truth values, thus we omit it.

Proposition 3.2.12. *Let* $\boldsymbol{X}$ *be a conditional set. Then:*

1. *If* $(\boldsymbol{x_n})$ *is a conditional sequence in* $\boldsymbol{X}$*, then* $[\![(\boldsymbol{x_n})\!\uparrow \text{ is a sequece in } \boldsymbol{X}\!\uparrow]\!] = I$*;*
2. *If* $(\boldsymbol{x_{n_k}})$ *is a conditional subsequence of* $(\boldsymbol{x_n})$*, then* $[\![(\boldsymbol{x_{n_k}})\!\uparrow \text{ is a subsequece of } (\boldsymbol{x_n})\!\uparrow]\!] = I$*.*

Proposition 3.2.13. *Let* $\boldsymbol{X}$ *be a conditional set,* $(\boldsymbol{x_j})_{\boldsymbol{j} \in \boldsymbol{J}}$ *be a conditional family of conditional elements of* $\boldsymbol{X}$*,* $\boldsymbol{\mathcal{F}}$ *a conditional collection of conditional subsets of* $\boldsymbol{X}$*.*

1. $(\boldsymbol{x_j})$ *is a conditional net in* $\boldsymbol{X}$ *if and only if* $[\![(\boldsymbol{x_j})\!\uparrow \text{ is a net in } \boldsymbol{X}\!\uparrow]\!] = I$*;*
2. $\boldsymbol{\mathcal{F}}$ *is a conditional filter base if and only if* $[\![\boldsymbol{\mathcal{F}}\!\Uparrow \textit{ is a filter base on } \boldsymbol{X}\!\uparrow]\!] = I$*;*
3. $\boldsymbol{\mathcal{F}}$ *is a conditional filter if and only if* $[\![\boldsymbol{\mathcal{F}}\!\Uparrow \textit{ is a filter on } \boldsymbol{X}\!\uparrow]\!] = I$*;*
4. $\boldsymbol{\mathcal{F}}$ *is a conditional ultrafilter if and only if* $[\![\boldsymbol{\mathcal{F}}\!\Uparrow \textit{ is a ultrafilter on } \boldsymbol{X}\!\uparrow]\!] = I$*.*

3.2.9 Conditional convergence

Definition 3.2.13. *Let* $(\boldsymbol{X},\boldsymbol{\mathcal{T}})$ *be a conditional topological space.*

- *A conditional net* $(\boldsymbol{x_j})_{\boldsymbol{j}\in\boldsymbol{J}}$ *in* $\boldsymbol{X}$ *conditionally converges to* $\boldsymbol{x}\in\boldsymbol{X}$ *if for any* $\boldsymbol{U}\in\boldsymbol{\mathcal{U}_x}$ *there exists* $\boldsymbol{j}_0\in\boldsymbol{J}$ *such that* $\boldsymbol{x_j}\in\boldsymbol{U}$ *for all* $\boldsymbol{j}\in\boldsymbol{J}$ *with* $\boldsymbol{j}\boldsymbol{\geq}\boldsymbol{j}_0$*. If* $(\boldsymbol{x_j})$ *conditionally converges exactly to one conditional element* $\boldsymbol{x}\in\boldsymbol{X}$*, then we write* $\mathbf{lim}_n\,\boldsymbol{x_j}=\boldsymbol{x}$*.*

- $\boldsymbol{x}\in\boldsymbol{X}$ *is a conditional cluster point of a conditional net* $(\boldsymbol{x_j})_{\boldsymbol{j}\in\boldsymbol{J}}$ *in* $\boldsymbol{X}$ *if for every* $\boldsymbol{U}\in\boldsymbol{\mathcal{U}_x}$ *and* $\boldsymbol{j}\in\boldsymbol{J}$*, there exists* $\boldsymbol{i}\in\boldsymbol{J}$ *with* $\boldsymbol{i}\boldsymbol{\geq}\boldsymbol{j}$ *such that* $\boldsymbol{x_i}\in\boldsymbol{U}$*.*

- *A conditional filter base* $\boldsymbol{\mathcal{G}}$ *on* $\boldsymbol{X}$ *conditionally converges to* $\boldsymbol{x}\in\boldsymbol{X}$*, if for every* $\boldsymbol{U}\in\boldsymbol{\mathcal{U}_x}$ *there exists* $\boldsymbol{B}\in\boldsymbol{\mathcal{G}}$ *such that* $\boldsymbol{B}\sqsubset\boldsymbol{U}$*.*

- $\boldsymbol{x}\in\boldsymbol{X}$ *is a conditional cluster point of a conditional filter base* $\boldsymbol{\mathcal{G}}$ *if for every* $\boldsymbol{U}\in\boldsymbol{\mathcal{G}}$ *and every* $\boldsymbol{V}\in\boldsymbol{\mathcal{U}_x}$ *it holds that* $\boldsymbol{U}\sqcap\boldsymbol{V}$ *is on* I*.*

The proof of the following result is omitted as it follows by an elementary manipulation of the Boolean truth values.

Proposition 3.2.14. *Let* $(\boldsymbol{X},\boldsymbol{\mathcal{T}})$ *be a conditional topological space,* $\boldsymbol{\chi}:=(\boldsymbol{x_j})_{\boldsymbol{j}\in\boldsymbol{J}}$ *a conditional net on* $\boldsymbol{X}$ *and* $\boldsymbol{\mathcal{G}}$ *a conditional filter base on* $\boldsymbol{X}$*. Then:*

- $\boldsymbol{\chi}$ *conditionally converges to* $\boldsymbol{x}\in\boldsymbol{X}$ *if and only if* $[\![\boldsymbol{\chi}\!\uparrow \text{ converges to } \boldsymbol{x}^{\bullet}]\!]=I$*;*
- $\boldsymbol{x}$ *is a conditional cluster point of* $\boldsymbol{\chi}$ *if and only if* $[\![\boldsymbol{x}^{\bullet} \text{ is a cluster point of } \boldsymbol{\chi}\!\uparrow]\!]=I$*;*
- $\boldsymbol{\mathcal{G}}$ *conditionally converges to* $\boldsymbol{x}\in\boldsymbol{X}$ *if and only if* $[\![\boldsymbol{\mathcal{G}}\!\Uparrow \text{ converges to } \boldsymbol{x}^{\bullet}]\!]=I$*;*
- $\boldsymbol{x}$ *is a conditional cluster point of* $\boldsymbol{\mathcal{G}}$ *if and only if* $[\![\boldsymbol{x}^{\bullet} \text{ is a cluster point of } \boldsymbol{\mathcal{G}}\!\Uparrow]\!]=I$*.*

3.2.10 Conditional compactness

Definition 3.2.14. *A conditional topological space* $(\boldsymbol{X},\boldsymbol{\mathcal{T}})$ *is said to be:*

- conditionally compact *if every conditional filter base* $\boldsymbol{\mathcal{G}}$ *on* $\boldsymbol{X}$ *has a conditional cluster point* $\boldsymbol{x}\in\boldsymbol{X}$*;*
- conditionally sequentially compact *if every conditional sequence* $(\boldsymbol{x_n})$ *in* $\boldsymbol{X}$ *admits a conditional subsequence* $(\boldsymbol{x_{n_k}})$ *which conditionally converges to some conditional element* $\boldsymbol{x}\in\boldsymbol{X}$*.*

We omit the proof of the following result because it follows by means of a simple manipulation of Boolean truth values.

Proposition 3.2.15. *Let* $(\boldsymbol{X},\boldsymbol{\mathcal{T}})$ *be a conditional topological space and* $\boldsymbol{Y}$ *a conditional subset of* $\boldsymbol{X}$*. Then:*

1. $\boldsymbol{Y}$ *is conditionally compact if and only if* $[\![\,\boldsymbol{Y}\!\uparrow \text{ is compact}]\!]=I$*;*
2. $\boldsymbol{Y}$ *is conditionally sequentially compact if and only if* $[\![\,\boldsymbol{Y}\!\uparrow \text{ is sequentially compact}]\!]=I$*.*

Comments 3.2.2. *Conditional compactness is defined differently in [32]; the authors introduce conditional compactness by using conditional open coverings. The definition that we provide is equivalent as shown in [32, Proposition 3.25]. We use conditional filter basis because we find them easier to interpret within* $\overline{V}^{(\mathcal{A})}$ *than conditional open coverings.*

3.2.11 Conditional real numbers

A conditional set $\mathbf{X}$ endowed with conditional operations $+ : \mathbf{X} \times \mathbf{X} \to \mathbf{X}$ and $\cdot : \mathbf{X} \times \mathbf{X} \to \mathbf{X}$ is called a *conditional commutative ring* if $(X, +, \cdot)$ is a commutative ring. We denote by $\mathbf{0}$ and $\mathbf{1}$ the conditional neutral elements for $+$ and $\cdot$, respectively. A conditional commutative ring $\mathbf{X}$ is called a *conditional field* if for every $\mathbf{x} \in \mathbf{X}$ such that $a_{x,0} = 0$ there exists $\mathbf{y} \in \mathbf{X}$ with $a_{y,0} = 0$ such that $\mathbf{xy} = \mathbf{1}$. A conditional field $\mathbf{X}$ is said to be a *conditionally ordered field* if there exists a conditional total order $\leq$ on $\mathbf{X}$ which satisfies that $\mathbf{x} \leq \mathbf{y}$ implies $\mathbf{x}+\mathbf{z} \leq \mathbf{y}+\mathbf{z}$ and $\mathbf{0} \leq \mathbf{x}, \mathbf{y}$ implies $\mathbf{0} \leq \mathbf{xy}$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{X}$.

In conditional set theory, the *conditional real numbers* $\mathbf{R}$ are constructed from the conditional rational numbers $\mathbf{Q}$ by adapting to the conditional setting the classical construction of the real numbers from Cauchy sequences of rational numbers. This process leads to a conditionally ordered field $(\mathbf{R}, +, \cdot, \leq)$ which is conditionally Dedekind complete.

We can consider the ascent $\mathbf{R}\uparrow$ of $\mathbf{R}$ as a conditional set. It is routine to verify that $\mathbf{R}\uparrow$ is a name for a Dedekind complete totally ordered field. In ZFC, any two Dedekind complete ordered fields are isomorphic (i.e. there exists an order preserving isomorphism of fields); that is, the set of real numbers is unique up to isomorphism. Thus, by the transfer principle, we can conclude that $\mathbf{R}\uparrow$ is a name for a representative of the set real numbers. As a consequence, we obtain the existence of a stable bijection between R and $\mathbb{R}_{\mathcal{A}}\downarrow$. This justifies the alternative definition of the ascent $\mathbf{R}\uparrow := \mathbb{R}_{\mathcal{A}}$ which we will adopt throughout.

Comments 3.2.3. *The Gordon's theorem on Boolean-valued real numbers (see Comments 2.1.1) asserts that, for any complete Boolean algebra $\mathcal{A}$, $\mathbb{R}_{\mathcal{A}}\downarrow$ is a universally complete Kantorovich space such that the complete Boolean algebra of band projections is isomorphic to $\mathcal{A}$ and, furthermore, each universally complete Kantorovich space with an order unity provides a name for the field of the real numbers when one considers the Boolean-valued model associated to the complete Boolean algebra of band projections.*

We can conclude that any universally complete Kantorovich space K with an order unity can be endowed with a conditioning

$$| : K \times \mathcal{A} \to \boldsymbol{R}$$

where $\boldsymbol{R}$ is the conditional set of real numbers associated to the complete Boolean algebra $\mathcal{A}$ of band projections of K.

3.2.12 Conditional metric spaces

We define $\mathbf{R}^{+}$ and $\mathbf{R}^{++}$ to be the conditional subsets of $\mathbf{R}$ generated by the stable subsets $R_{+} := \{r \in R \colon r \geq 0\}$ and $R_{++} := \{r \in R \colon r > 0\}$, respectively.

Definition 3.2.15. *A conditional function $\boldsymbol{d} \colon \boldsymbol{X} \times \boldsymbol{X} \to \boldsymbol{R}^{+}$ is called a* conditional metric *if:*

1. $\boldsymbol{d}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{0}$ *if, and only if,* $\boldsymbol{x} = \boldsymbol{y}$*;*
2. $\boldsymbol{d}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{d}(\boldsymbol{y}, \boldsymbol{x})$ *for all* $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$*;*
3. $\boldsymbol{d}(\boldsymbol{x}, \boldsymbol{z}) \leq \boldsymbol{d}(\boldsymbol{x}, \boldsymbol{y}) + \boldsymbol{d}(\boldsymbol{y}, \boldsymbol{z})$ *for all* $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \boldsymbol{X}$.

The pair $(\boldsymbol{X}, \boldsymbol{d})$ is called a conditional metric space.

Given a conditional metric space $(\mathbf{X}, \mathbf{d})$, for $\mathbf{x} \in \mathbf{X}$ and $\mathbf{r} \in \mathbf{R}^{++}$, we define the *conditional ball*, denoted by $\mathbf{B_r}(\mathbf{x})$ to be the conditional subset of $\mathbf{X}$ generated by the stable subset

$$B_r(x) := \{y \in X \colon d(x, y) < r\}.$$

The conditional collection

$$\boldsymbol{\mathcal{B}} := \{\mathbf{B_r}(\mathbf{x})|a\colon \mathbf{x} \in \mathbf{X},\ \mathbf{r} \in \mathbf{R}^{++},\ a \in \mathcal{A}\}$$

is a conditional base for a conditional topology $\boldsymbol{\mathcal{T}}_{\mathbf{d}}$.

Definition 3.2.16. *Let $(\boldsymbol{X}, \boldsymbol{d})$ be a conditional metric space.*

- *A conditional sequence $(\boldsymbol{x_n})$ in $\boldsymbol{X}$ is* conditionally Cauchy *if for every $\boldsymbol{r} \in \boldsymbol{R}^{++}$ there exists $\boldsymbol{n_r} \in \boldsymbol{N}$ such that*

$$\boldsymbol{d}(\boldsymbol{x_p}, \boldsymbol{x_q}) \leq \boldsymbol{r} \quad \text{for all } \boldsymbol{p}, \boldsymbol{q} \geq \boldsymbol{n_r},\ \boldsymbol{p}, \boldsymbol{q} \in \boldsymbol{N}.$$

- *$(\boldsymbol{X}, \boldsymbol{d})$ is* conditionally complete *if every conditionally Cauchy sequence conditionally converges to some conditional element $\boldsymbol{x} \in \boldsymbol{X}$.*

We omit the proof of the following result because it follows by means of a simple manipulation of Boolean truth values.

Proposition 3.2.16. *Let $\boldsymbol{d}\colon \boldsymbol{X} \times \boldsymbol{X} \to \boldsymbol{R}^{+}$ be a conditional metric. Then there exists a name $\boldsymbol{d}\!\uparrow$ for a metric on $X\!\uparrow$ such that*

$$[\![\boldsymbol{d}\!\uparrow(\boldsymbol{x}^{\bullet}, \boldsymbol{y}^{\bullet}) = \boldsymbol{d}(\boldsymbol{x}, \boldsymbol{y})^{\bullet}]\!] = I \quad \text{for all } \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}.$$

In addition, $\boldsymbol{\mathcal{T}}_{\boldsymbol{d}}\!\Uparrow$ is a name for the set of non-empty open sets w.r.t. the topology induced by $\boldsymbol{d}\!\uparrow$.

Moreover,

1. *A conditional sequence $(\boldsymbol{x_n})$ is conditionally Cauchy if and only if $[\![(\boldsymbol{x_n})\!\uparrow \text{ is Cauchy}]\!] = I$;*
2. *$(\boldsymbol{X}, \boldsymbol{d})$ is conditionally complete if and only if $[\![(\boldsymbol{X}\!\uparrow, \boldsymbol{d}\!\uparrow)_{\mathcal{A}} \text{ is complete}]\!] = I$.*

3.2.13 Conditional vector spaces

Definition 3.2.17. *A conditional set $\boldsymbol{X}$ endowed with conditional operations $\boldsymbol{+}\colon \boldsymbol{X} \times \boldsymbol{X} \to \boldsymbol{X}$ and $\boldsymbol{\cdot}\colon \boldsymbol{R} \times \boldsymbol{X} \to \boldsymbol{X}$ is called a* conditional vector space *if $(X, +, \cdot)$ is an R-module.*

Definition 3.2.18. *Let $\boldsymbol{X}$ be a conditional vector space and $\boldsymbol{Y}$ a conditional subset of $\boldsymbol{X}$. We say that $\boldsymbol{Y}$ is:*

1. *a* conditional subspace *if $\boldsymbol{Y} \boldsymbol{+} \boldsymbol{Y} \sqsubset \boldsymbol{Y}$ and $\boldsymbol{r}\boldsymbol{Y} \sqsubset \boldsymbol{Y}$ for all $\boldsymbol{r} \in \boldsymbol{R}$;*
2. conditionally convex *if $\boldsymbol{r}\boldsymbol{Y} \boldsymbol{+} (\boldsymbol{1} - \boldsymbol{r})\boldsymbol{Y} \sqsubset \boldsymbol{Y}$ for all $\boldsymbol{r} \in \boldsymbol{R}$ with $\boldsymbol{0} \leq \boldsymbol{r} \leq \boldsymbol{1}$;*
3. conditionally absorbing *if for every $\boldsymbol{x} \in \boldsymbol{X}$ there exists $\boldsymbol{r} \in \boldsymbol{R}^{++}$ such that $\boldsymbol{x} \in \boldsymbol{r}\boldsymbol{Y}$;*
4. conditionally balanced *if $\boldsymbol{r}\boldsymbol{x} \in \boldsymbol{Y}$ for every $\boldsymbol{x} \in \boldsymbol{Y}$ and every $\boldsymbol{r} \in \boldsymbol{R}$ with $|\boldsymbol{r}| \leq \boldsymbol{1}$.*

Definition 3.2.19. *Let $\boldsymbol{X}$ be a conditional vector space.*

- *Given a conditional subset $\boldsymbol{Y}$ of $\boldsymbol{X}$ we define its conditional span to be*

$$\mathbf{span}_{\boldsymbol{R}}\boldsymbol{Y} := \bigsqcap\{\boldsymbol{F} \sqsubset \boldsymbol{X}\colon \boldsymbol{F} \text{ is a conditional subspace with } \boldsymbol{Y} \sqsubset \boldsymbol{F}\}.$$

- *$\boldsymbol{X}$ is said to be* conditionally finitely generated *if there exists a conditionally finite subset $\boldsymbol{Y}$ of $\boldsymbol{X}$ such that $\mathbf{span}_{\boldsymbol{R}}\boldsymbol{Y} = \boldsymbol{X}$.*

Definition 3.2.20. *A conditional function* $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ *between conditional vector spaces is* conditionally linear *if* $f : X \to Y$ *is an R-module homomorphism.*

Definition 3.2.21. *A conditional function* $\| \cdot \| : \boldsymbol{X} \to \boldsymbol{R}^+$ *is a* conditional seminorm *on* $\boldsymbol{X}$ *if:*

1. $\| \boldsymbol{r}\boldsymbol{x} \| = | \boldsymbol{r} | \, \| \boldsymbol{x} \|$ *for all* $\boldsymbol{r} \in \boldsymbol{R}$ *and* $\boldsymbol{x} \in \boldsymbol{X}$;
2. $\| \boldsymbol{x} + \boldsymbol{y} \| \leq \| \boldsymbol{x} \| + \| \boldsymbol{y} \|$ *for all* $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$.

If moreover, $\| \boldsymbol{x} \| = \boldsymbol{0}$ *implies* $\boldsymbol{x} = \boldsymbol{0}$*, then* $\| \cdot \|$ *is a* conditional norm *on* $\boldsymbol{X}$.

Definition 3.2.22. *A pair* $(\boldsymbol{X}, \boldsymbol{\mathcal{T}})$ *is a* conditionally locally convex space *if* $\boldsymbol{X}$ *is a conditional vector space,* $\boldsymbol{\mathcal{T}}$ *is a conditional topology such that* $\boldsymbol{+} : \boldsymbol{X} \times \boldsymbol{X} \to \boldsymbol{X}$ *and* $\boldsymbol{\cdot} : \boldsymbol{R} \times \boldsymbol{X} \to \boldsymbol{X}$ *are conditionally continuous, and there exists a conditional neighborhood base* $\boldsymbol{\mathcal{U}}$ *of* $\boldsymbol{0}$ *consisting of conditionally convex sets.*

Definition 3.2.23. *Let* $\boldsymbol{\mathcal{P}}$ *be a conditional collection of conditional seminorms on a conditional vector space* $\boldsymbol{X}$*. For any conditionally finite subset* $\boldsymbol{F}$ *of* $\boldsymbol{\mathcal{P}}$ *and* $\boldsymbol{r} \in \boldsymbol{R}^{++}$*, we define* $\boldsymbol{U_{F,r}}$ *the conditional subset of* $\boldsymbol{X}$ *generated by the stable set*

$$U_{F,r} := \{x \in X : \|x\| < r \text{ for all } \| \cdot \| \in F\}.$$

The conditional collection

$$\boldsymbol{\mathcal{U}} = \{ \boldsymbol{U_{F,r}} | a : \boldsymbol{r} \in \boldsymbol{R}^{++},\ \boldsymbol{F} \sqsubset \boldsymbol{\mathcal{P}} \text{ conditionally finite, } a \in \mathcal{A}\}.$$

is a conditional neighborhood base of $\boldsymbol{0} \in \boldsymbol{X}$ *for a conditionally locally convex topology on* $\boldsymbol{X}$ *which is called the conditional weak topology conditionally induced by* $\boldsymbol{\mathcal{P}}$ *and is denoted by* $\langle \boldsymbol{\mathcal{P}} \rangle$.

An example of a conditional locally convex space is a conditionally normed space. If $(\mathbf{X}, \| \cdot \|)$ is a conditionally normed space, then the conditional norm defines a conditional metric $\mathbf{d}(\mathbf{x}, \mathbf{y}) := \| \mathbf{x} - \mathbf{y} \|$, which induces a conditional topology. A conditionally normed space which is conditionally complete is called a *conditional Banach space.* For instance, $(\mathbf{R}, | \cdot |)$ is a conditional Banach space.

If $(\mathbf{X}, \boldsymbol{\mathcal{T}}_{\mathbf{X}})$ and $(\mathbf{X}, \boldsymbol{\mathcal{T}}_{\mathbf{Y}})$ are conditionally locally convex spaces, then we define $\mathbf{B}(\mathbf{X}, \mathbf{Y})$ the conditional sets generated by the stable subset

$$\mathbf{B}(\mathbf{X}, \mathbf{Y}) := \{\mathbf{f}|a : \mathbf{f} \in \mathbf{fun}(\mathbf{X}, \mathbf{Y}),\ \mathbf{f} \text{ is conditionally linear continuous, } a \in \mathcal{A}\}.$$

In particular, if $\mathbf{Y} = \mathbf{R}$, we define the *conditional dual space* of $\mathbf{X}$ to be $\mathbf{E}^* := \mathbf{B}(\mathbf{X}, \mathbf{Y})$.

If $\mathbf{X}$, $\mathbf{Y}$ are conditionally normed spaces, for any $\mathbf{f} \in \mathbf{B}(\mathbf{X}, \mathbf{Y})$ we define

$$\|f\| := \sup\{|f(x)| : x \in X,\ \|x\| \leq 1\},$$

Then, $f \mapsto \|f\|$ is a stable function. The corresponding conditional function $\| \cdot \|$ is a conditional norm on $\mathbf{B}(\mathbf{X}, \mathbf{Y})$.

Let us see how the notions above can be interpreted within $V^{(\mathcal{A})}$. The following proposition can be proven by means of an elementary manipulation of the Boolean truth values.

Proposition 3.2.17. *Let* $\boldsymbol{X}$ *is a conditional vector space. Then* $\boldsymbol{X}{\uparrow}$ *is name for a vector space with* $[\![\boldsymbol{x}^\bullet + \boldsymbol{y}^\bullet = (\boldsymbol{x} + \boldsymbol{y})^\bullet]\!] = I$ *and* $[\![\boldsymbol{r}^\bullet \boldsymbol{x}^\bullet = (\boldsymbol{r}\boldsymbol{x})^\bullet]\!] = I$ *for all* $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$ *and* $\boldsymbol{r} \in \boldsymbol{R}$*. If* $\boldsymbol{Y}$ *is a conditional subset, then:*

1. $\boldsymbol{Y}$ *is a conditionally subspace of* $\boldsymbol{X}$ *if and only if* $[\![\boldsymbol{Y}{\uparrow} \text{ is a subspace of } \boldsymbol{X}{\uparrow}]\!] = I$;

2. $\boldsymbol{Y}$ *is a conditionally convex if and only if* $[\![\, \boldsymbol{Y}\!\uparrow \text{ is convex}]\!] = I$;

3. $\boldsymbol{Y}$ *is a conditionally absorbing if and only if* $[\![\, \boldsymbol{Y}\!\uparrow \text{ is absorbing}]\!] = I$;

4. $\boldsymbol{Y}$ *is a conditionally balanced if and only if* $[\![\, \boldsymbol{Y}\!\uparrow \text{ is balanced}]\!] = I$;

5. $[\![(\mathbf{span}_{\boldsymbol{R}}\, \boldsymbol{Y})\!\uparrow = \operatorname{span}_{\mathbb{R}}(\boldsymbol{Y}\!\uparrow)]\!] = I$.

6. $\boldsymbol{X}$ *is conditionally finitely generated if and only if* $[\![\boldsymbol{X}\!\uparrow$ *is finitally generated*$]\!] = I$.

Proposition 3.2.18. *If* $(\boldsymbol{X}, \|\cdot\|)$ *is a conditionally normed space, then* $(\boldsymbol{X}\!\uparrow, \|\cdot\|\!\uparrow)_{\mathcal{A}}$ *is a name for a normed space. Moreover,* $[\![\langle\, \|\cdot\|\!\uparrow\rangle \cup\{\emptyset\} = \langle\|\cdot\|\!\uparrow\rangle]\!] = I$ *and* $(\boldsymbol{X}, \|\cdot\|)$ *is conditionally Banach if and only if* $[\![(\boldsymbol{X}\!\uparrow, \|\cdot\|\!\uparrow)_{\mathcal{A}} \text{ is Banach}]\!] = I$.

Proposition 3.2.19. *If* $(\boldsymbol{X}, \boldsymbol{\mathcal{T}})$ *is a* conditionally locally convex space*, then there exists a name* $\mathbb{T}$ *for a locally convex topology on* $\boldsymbol{X}\!\uparrow$ *such that* $[\![\boldsymbol{\mathcal{T}}\!\Uparrow \cup \{\emptyset\} = \mathbb{T}]\!] = I$.

Moreover if $\boldsymbol{\mathcal{P}}$ *is a conditional collection of conditional seminorms on* $\boldsymbol{X}$ *such that* $\boldsymbol{\mathcal{T}} = \langle\, \boldsymbol{\mathcal{P}}\,\rangle$*, then* $[\![\mathbb{T} = \langle \boldsymbol{\mathcal{P}}\!\uparrow\rangle]\!] = I$.

Proposition 3.2.20. *If* $(\boldsymbol{X}, \boldsymbol{\mathcal{T}}_{\boldsymbol{X}})$, $(\boldsymbol{Y}, \boldsymbol{\mathcal{T}}_{\boldsymbol{Y}})$ *are conditional locally convex spaces, then*

$$[\![\boldsymbol{B}(\boldsymbol{X}, \boldsymbol{Y})\!\uparrow = B(\boldsymbol{X}\!\uparrow, \boldsymbol{Y}\!\uparrow)]\!] = I.$$

In particular, $[\![\boldsymbol{X}^*\!\uparrow = (\boldsymbol{X}\!\uparrow)^*]\!] = I$.

3.2.14 Conditional dual pairs

Definition 3.2.24. *Let* $\boldsymbol{X}$, $\boldsymbol{Y}$ *be conditional vector spaces. The pair* $(\boldsymbol{X}, \boldsymbol{Y})$ *is called a* conditional dual pair *with respect to the conditional bi-linear function*

$$\langle\cdot,\cdot\rangle : \boldsymbol{X} \times \boldsymbol{Y} \longrightarrow \boldsymbol{R}$$

(i.e. both $\langle\cdot, \boldsymbol{y}\rangle : \boldsymbol{X} \to \boldsymbol{R}$ *and* $\langle \boldsymbol{x}, \cdot\rangle : \boldsymbol{Y} \to \boldsymbol{R}$ *are conditionally linear) if the following conditions are fulfilled:*

1. $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \boldsymbol{0}$ *for all* $\boldsymbol{x} \in \boldsymbol{X}$ *implies* $\boldsymbol{y} = \boldsymbol{0}$;

2. $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \boldsymbol{0}$ *for all* $\boldsymbol{y} \in \boldsymbol{Y}$ *implies* $\boldsymbol{x} = \boldsymbol{0}$.

We will say that $\langle \boldsymbol{X}, \boldsymbol{Y}\rangle$ *is a conditional dual pair.*

Definition 3.2.25. *Let* $\langle \boldsymbol{X}, \boldsymbol{Y}\rangle$ *be a conditional dual pair. For any* $\boldsymbol{y} \in \boldsymbol{Y}$*, the conditional function*

$$\|\cdot\|_{\boldsymbol{y}} : \boldsymbol{X} \to \boldsymbol{R}^+, \quad \|\,\boldsymbol{x}\,\|_{\boldsymbol{y}} := |\,\langle\, \boldsymbol{x}, \boldsymbol{y}\,\rangle\,|$$

is a conditional seminorm. The conditional collection of conditional seminorms generated by $\{\|\cdot\|_y : y \in Y\}$ *conditionally induces a conditionally locally convex topology on* $\boldsymbol{X}$ *which is called the* conditionally weak topology *induced by* $\langle \boldsymbol{X}, \boldsymbol{Y}\rangle$ *and is denoted by* $\boldsymbol{\sigma}(\boldsymbol{X}, \boldsymbol{Y})$.

We do not prove the following propositions because they follow by elementary manipulations of the Boolean truth values.

Proposition 3.2.21. *Let* $\langle \boldsymbol{X}, \boldsymbol{Y}\rangle$ *be a conditional dual pair. Then there exists a name* $\langle \boldsymbol{X}\!\uparrow, \boldsymbol{Y}\!\uparrow\rangle_{\mathcal{A}}$ *for a dual pair such that*

$$[\![\langle \boldsymbol{x}^\bullet, \boldsymbol{y}^\bullet\rangle = \langle\, \boldsymbol{x}, \boldsymbol{y}\,\rangle^\bullet]\!] = I \quad \text{for all } \boldsymbol{x} \in \boldsymbol{X},\ \boldsymbol{y} \in \boldsymbol{Y}.$$

Moreover, $[\![\boldsymbol{\sigma}(\boldsymbol{X}, \boldsymbol{Y})\!\Uparrow \cup \{\emptyset\} = \sigma(\boldsymbol{X}\!\uparrow, \boldsymbol{Y}\!\uparrow)]\!] = I$.

Corollary 3.2.2. *Let* $(\boldsymbol{X}, \boldsymbol{\mathcal{T}})$ *be a conditional locally convex space, then*

$$[\![\boldsymbol{\sigma}(\boldsymbol{X}, \boldsymbol{X}^*)\!\Uparrow \cup \{\emptyset\} = \sigma(\boldsymbol{X}\!\uparrow, \boldsymbol{X}\!\uparrow^*)]\!] = I \quad \text{and} \quad [\![\boldsymbol{\sigma}(\boldsymbol{X}^*, \boldsymbol{X})\!\Uparrow \cup \{\emptyset\} = \sigma(\boldsymbol{X}\!\uparrow^*, \boldsymbol{X}\!\uparrow)]\!] = I.$$

Comments 3.2.4. *Conditional dual pairs are introduced and discussed in [32]. A more detailed discussion can be found in [73, Chapter 4].*

3.2.15 Conditional extended real numbers

In conditional set theory it is not introduced a conditional analogue of the extended real numbers $\overline{\mathbb{R}}$. We can construct them by means of a conditional order completion of $\mathbf{R}$ as follows:

Consider the collection of ordered sextuples $(-\infty, r, \infty, a, b, c)$ with $r \in R$ and $(a, b, c) \in p(I)$. We define $\overline{R}$ to be the quotient set of the equivalence relation $\sim$ defined by

$$(-\infty, r, \infty, a, b, c) \sim (-\infty, r', \infty, a', b', c') \quad \text{whenever } a = a',\ b = b',\ c = c' \text{ and } r|b = r'|b'.$$

Denote by $[-\infty, \infty, r, a, b, c]$ the class of $(-\infty, \infty, r, a, b, c)$.

Now, we consider the equivalence relation $\sim$ on $\overline{R} \times \mathcal{A}$ defined by

$$([-\infty, r, \infty, a, b, c], d) \sim ([-\infty, r', \infty, a', b', c'], d')$$

$$\text{whenever}$$

$$d = d',\ a \wedge d = a' \wedge d',\ b \wedge d = b' \wedge d',\ c \wedge d = c' \wedge d' \text{ and } r|b \wedge d = r'|b' \wedge d'.$$

Then the quotient set, say $\overline{\mathbf{R}}$, is a conditional set which we will refer to as the *conditional extended real numbers.*

Note that (C1) and (C2) are clear from the definition of $\overline{\mathbf{R}}$. If $([-\infty, r_i, \infty, a_i, b_i, c_i])_i$ is a family in $\overline{R}$ and $(d_i) \in p(I)$, we define

$$\sum [-\infty, r_i, \infty, a_i, b_i, c_i]|d_i = [-\infty, r, \infty, a, b, c],$$

where $a := \bigvee_i d_i \wedge a_i$, $a := \bigvee_i d_i \wedge a_i$, $b := \bigvee_i d_i \wedge b_i$, $c := \bigvee_i d_i \wedge c_i$ and $r := \sum r_i|d_i \wedge b_i + 0|b^c$.

Notice that for any $[-\infty, r, \infty, a, b, c] \in \overline{R}$, one has

$$[-\infty, r, \infty, a, b, c] = [-\infty, 0, \infty, I, 0, 0]|a + [-\infty, r, \infty, 0, I, 0]|b + [-\infty, 0, \infty, I, 0, 0]|c.$$

In view of this identity, we will use the notation

$$-\infty|a + r|b + \infty|c := [-\infty, r, \infty, a, b, c]$$

for the elements of $\overline{R}$ and the notation $\boldsymbol{-\infty}|a + \mathbf{r}|b + \boldsymbol{\infty}|c$ for their corresponding conditional elements.

In addition we can extend the conditional total order of $\mathbf{R}$ to $\overline{R}$ as follows:

$$\boldsymbol{-\infty}|a + \mathbf{r}|b + \boldsymbol{\infty}|c \boldsymbol{\leq} \boldsymbol{-\infty}|a' + \mathbf{r}'|b' + \boldsymbol{\infty}|c' \quad \text{if } a \geq a',\ c \leq c',\ \mathbf{r}|b \wedge b' \boldsymbol{\leq} \mathbf{r}'|b \wedge b'.$$

Let $\mathbf{E}$ be a conditional vector space and $\mathbf{f} : \mathbf{E} \to \overline{\mathbf{R}}$ a conditional function.

- $\mathbf{f}$ is said to be *conditionally proper* if $\mathbf{f}(\mathbf{x}) \boldsymbol{>} \boldsymbol{-\infty}$ for all $\mathbf{x} \in \mathbf{E}$ and there exists some $\mathbf{x} \in \mathbf{E}$ such that $\mathbf{f}(\mathbf{x}) \in \mathbf{R}$.
- If $\mathbf{f}$ is conditionally proper, the sets

$$\operatorname{dom}(f) := \{x \in E \colon f(x) \in R\},$$

$$\operatorname{epi}(f) := \{(x, r) \in E \times R \colon f(x) \leq r\},$$

are stable and the corresponding conditional sets, denoted by $\mathbf{dom}(\mathbf{f})$ and $\mathbf{epi}(\mathbf{f})$, are called the *conditional effective domain* and *conditional epigraph* of $\mathbf{f}$, respectively.

- Let $\mathbf{r} \in \mathbf{R}$ be such that
$$V_r(f) := \{x \in E\colon f(x) \leq r\}$$
is non-empty. Then $V_r(f)$ is stable and the corresponding conditional set, denoted by $\mathbf{V_r}(\mathbf{f})$, is called the *conditional* $\boldsymbol{r}$*-sub-level set* of $\mathbf{f}$.

- A conditionally proper function $\mathbf{f}: \mathbf{E} \to \overline{\mathbf{R}}$ is said to be *conditionally convex* if
$$\mathbf{f}(\mathbf{rx} + (\mathbf{1} - \mathbf{r})\mathbf{y}) \leq \mathbf{rf}(\mathbf{x}) + (\mathbf{1} - \mathbf{r})\mathbf{f}(\mathbf{y}) \quad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbf{E} \text{ and } \mathbf{r} \in \mathbf{R} \text{ with } \mathbf{0} \leq \mathbf{r} \leq \mathbf{1}.$$

Next, we will define some ascent operations:

We start by defining the ascent of $\overline{\mathbf{R}}$ to be $\overline{\mathbf{R}}\!\uparrow := \overline{\mathbb{R}}_{\mathcal{A}}$.

Due to the mixing principle we have the following:

Proposition 3.2.22. *The mapping*
$$-\infty|a + r|b + \infty|c \longmapsto -\breve{\infty}a + \boldsymbol{r}^{\bullet}b + \breve{\infty}c$$
is a stable bijection between $\overline{R}$ *and* $\overline{\mathbb{R}}_{\mathcal{A}}$.

The Proposition above defines a canonical bijection:
$$\begin{array}{cccc} i_{\overline{R}}: & \overline{R} & \longrightarrow & \overline{\mathbb{R}}_{\mathcal{A}}\!\downarrow \\ & r & \longmapsto & \mathbf{r}^{\bullet} \\ & \mathbb{r}^{\circ} & \longleftarrow & \mathbb{r} \end{array}$$

Let $\mathbf{E}$ be a conditional vector space and $\mathbf{f}: \mathbf{E} \to \overline{\mathbf{R}}$ is a conditional function. The mapping
$$\mathbf{E}\!\uparrow \longrightarrow \overline{\mathbb{R}}_{\mathcal{A}} \quad \mathbf{x}^{\bullet} \mapsto \mathbf{f}(\mathbf{x})^{\bullet}$$
is extensional, then we can define the corresponding ascent $\mathbf{f}\!\uparrow$.

We define the ascent $\mathbf{dom}(\mathbf{f})\!\uparrow$ to be the ascent of $\mathbf{dom}(\mathbf{f})$ as a conditional subset of $\mathbf{E}$.

We define the ascent $\mathbf{epi}(\mathbf{f})\!\uparrow$ as the ascent of $\mathbf{epi}(\mathbf{f})$ a conditional subset of the conditional product $\mathbf{E} \times \mathbf{R}$.

We do not prove the following propositions because they follow by elementary manipulations of the Boolean truth values.

Proposition 3.2.23. *Let* $\boldsymbol{E}$ *be a conditional vector space and* $\boldsymbol{f}: \boldsymbol{E} \to \overline{\boldsymbol{R}}$ *is a conditional function. The following holds:*

1. $[\![\mathbf{dom}(\boldsymbol{f})\!\uparrow = \mathrm{dom}(\boldsymbol{f}\!\uparrow)]\!] = I$ *and* $[\![\mathbf{epi}(\boldsymbol{f})\!\uparrow = \mathrm{epi}(\boldsymbol{f}\!\uparrow)]\!] = I$;
2. $\boldsymbol{f}$ *is conditionally proper if and only if* $[\![\boldsymbol{f}\!\uparrow \text{ is proper}]\!] = I$. *In this case,* $\boldsymbol{f}$ *is conditionally convex if and only if* $[\![\boldsymbol{f}\!\uparrow \text{ is convex}]\!] = I$.
3. *If* $V_r(f) \neq \emptyset$ *with* $\boldsymbol{r} \in \boldsymbol{R}$, *then* $[\![(\boldsymbol{V_r}(\boldsymbol{f}))\!\uparrow = V_{\boldsymbol{r}^{\bullet}}(\boldsymbol{f}\!\uparrow)]\!] = I$.

Suppose that $\mathbf{X}$ is a conditional subset and $\mathbf{f}: \mathbf{X} \to \overline{\mathbf{R}}$ is a conditional function. We denote by $\mathbf{sup}_{\mathbf{x} \in \mathbf{X}} \mathbf{f}(\mathbf{x})$ (resp. $\mathbf{inf}_{\mathbf{x} \in \mathbf{X}} \mathbf{f}(\mathbf{x})$) the conditional supremum (resp. infimum) of the conditional image $\mathbf{f}(\mathbf{X})$.

Proposition 3.2.24. *Let* $\boldsymbol{E}$ *be a conditional vector space,* $\boldsymbol{Y}$ *a conditional subset of* $\boldsymbol{E}$, *and* $\boldsymbol{f}\colon \boldsymbol{Y} \to \overline{\boldsymbol{R}}$ *is a conditional function. Then*
$$\left[\!\!\left[\left(\mathbf{sup}_{\boldsymbol{y} \in \boldsymbol{Y}} \boldsymbol{f}(\boldsymbol{y})\right)^{\bullet} = \sup_{y \in \boldsymbol{Y}\uparrow} \boldsymbol{f}\!\uparrow\!(y)\right]\!\!\right] = I \quad \text{and} \quad \left[\!\!\left[\left(\mathbf{inf}_{\boldsymbol{y} \in \boldsymbol{Y}} \boldsymbol{f}(\boldsymbol{y})\right)^{\bullet} = \inf_{y \in \boldsymbol{Y}\uparrow} \boldsymbol{f}\!\uparrow\!(y)\right]\!\!\right] = I.$$

3.3 Instances of application of the Boolean-valued transfer principle

Once we have shown how the different elements of conditional set theory are interpreted within the Boolean-valued universe, we will see how any conditional version of any theorem in terms of the objects listed before holds by means of the Boolean-valued principle. In general, we call that a proof of a result is *step-by-step* if it is carried out by conventional means, that is, it is not derived from a transfer principle. The reader interested in step-by-step proofs the results below can find the some of them in Appendix B.

We start with a conditional version of the Schauder–Tychonov fixed point theorem:

Theorem 3.3.1. *Let $(\boldsymbol{X}, \boldsymbol{\mathcal{T}})$ be a conditional locally convex space which is conditionally Hausdorff. If $\boldsymbol{C}$ is a conditionally compact conditional subset of $\boldsymbol{X}$ and $\boldsymbol{f}\colon \boldsymbol{C} \to \boldsymbol{C}$ is a conditionally continuous function, then there exists $\boldsymbol{x}$ in $\boldsymbol{C}$ such that $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{x}$.*

We can consider the names $\mathbf{X}\!\uparrow$, $\boldsymbol{\mathcal{T}}\!\Uparrow$, $\mathbf{C}\!\uparrow$, and $\mathbf{f}\!\uparrow$. Then $\boldsymbol{\mathcal{T}}\!\Uparrow$ is a name for the set of non-empty open sets of a locally convex topology on $\mathbf{X}\!\uparrow$, $\mathbf{C}\!\uparrow$ is a name for a compact subset of $\mathbf{X}\!\uparrow$, and $\mathbf{f}\!\uparrow$ is a name for a continuous function from $\mathbf{C}\!\uparrow$ to $\mathbf{C}\!\uparrow$. If T denotes the statement of the Schauder–Tychonov fixed point theorem, then the statement above, say $\mathbf{T}$, is nothing else but a reformulation of the theorem "$[\![T]\!] = I$", which holds due to the Boolean-valued transfer principle, so no proof is needed.

Let us see another example. Recall the theorem of the open map:

> "If $(X, |\cdot|)$ and $(Y, |\cdot|)$ are Banach spaces and $\lambda : X \to Y$ is a surjective continuous linear function, then $\lambda(O)$ is open whenever O is open."

Then we have the following:

Theorem 3.3.2. *Let $(\boldsymbol{X}, \|\cdot\|)$ and $(\boldsymbol{Y}, \|\cdot\|)$ be conditional Banach spaces and $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ a conditionally surjective continuous function, then $\boldsymbol{f}(\boldsymbol{O})$ is conditionally open whenever $\boldsymbol{O}$ is conditionally open.*

Again, what we have above is a reformulation of the theorem "$[\![$open map theorem$]\!] = I$". It holds due to the Boolean-valued transfer principle.

Finally, recall the Heine-Borel theorem:

> "Let $(X, |\cdot|)$ be a finite-dimensional normed space, then a subset C of X is compact if and only if C is closed and norm bounded."

The transfer principle yields the following:

Theorem 3.3.3. *Let $(\boldsymbol{E}, \|\cdot\|)$ be a conditionally normed space which is conditionally finitely generated. A conditional subset $\boldsymbol{Y}$ of $\boldsymbol{E}$ is conditionally compact if and only if $\boldsymbol{Y}$ is conditionally close and conditionally norm bounded.*

Remark 3.3.1. *A conditional version of the Heine-Borel theorem was obtained in [65, Theorem 5.5.8].*

Clearly, the number of instances of application can be easily increased. The same methodology applies to the different conditional versions of classical theorems proven in [32].

The next theorems were proven by conventional means in the first stage of the research period preceding this thesis. They can also be derived from the Boolean-valued transfer principle applied to their classical versions. The reader can find step-by-step proof of them in the Appendix B.

We have a conditional version of the Baire category theorem:

Theorem 3.3.4. *[Conditional version of Baire Category theorem] Suppose that a conditionally complete metric space* $(\boldsymbol{E}, \boldsymbol{d})$ *is the conditional union of a conditionally countable family* $(\boldsymbol{E_n})_{\boldsymbol{n}\in\boldsymbol{N}}$ *of conditionally closed subsets, then there exist a conditional ball* $\boldsymbol{B_r}(\boldsymbol{x})$ *and* $\boldsymbol{n} \in \boldsymbol{N}$ *such that* $\boldsymbol{B_r}(\boldsymbol{x}) \sqsubset \boldsymbol{E_n}$*.*

Next, we present a conditional analogue of the Uniform Boundedness principle:

Theorem 3.3.5. *[Conditional version of the Uniform Boundedness Principle] Let* $\boldsymbol{E}$ *be a conditional Banach space, and let* $\boldsymbol{F}$ *be a conditionally normed space. Suppose that* $\boldsymbol{S}$ *is a conditional subset of* $\boldsymbol{B}(\boldsymbol{E}, \boldsymbol{F})$*, such that for every* $\boldsymbol{x} \in \boldsymbol{E}$ *it is satisfied that* $\sup_{\boldsymbol{T}\in\boldsymbol{S}} \| \boldsymbol{T}(\boldsymbol{x}) \| \in \boldsymbol{R}$*. Then* $\sup_{\boldsymbol{T}\in\boldsymbol{S}} \| \boldsymbol{T} \| \in \boldsymbol{R}$*.*

The following is a conditional version of the Eberlein-Šmulian theorem:

Theorem 3.3.6. *[Conditional Eberlein-Šmulian theorem] A conditional subset of a conditionally normed space* $\boldsymbol{E}$ *is conditionally weakly compact if, and only if, it is conditionally weakly sequentially compact.*

Recall that a Banach space $(X, |\cdot|)$ is weakly compactly generated if there exists a subset $C \subset X$ which is weakly compact, convex and balanced such that $X = \mathrm{span}_{\mathbb{R}} C$. The Amir-Lindenstrauss theorem asserts that every weakly compactly generated Banach space has a weak-$*$ sequentially compact dual unit ball.

We introduce the conditional analogue of the notion of weakly compactly generated space.

Definition 3.3.1. *A conditional Banach space* $(\boldsymbol{E}, \|\cdot\|)$ *is conditionally weakly compactly generated if there exists a conditional subset* $\boldsymbol{C}$ *of* $\boldsymbol{E}$ *which is conditionally weakly compact, conditionally convex and conditionally balanced such that* $\boldsymbol{E} = \mathbf{span}_{\boldsymbol{R}} \boldsymbol{C}$*.*

Clearly, a conditional Banach space $\mathbf{E}$ is conditionally weakly compactly generated if and only if $\mathbf{E}\!\uparrow$ is a name for a weakly compactly generated Banach space. Thus the transfer principle applied to the classical Amir-Lindenstrauss theorem proves the following:

Theorem 3.3.7. *[Conditional Amir-Lindenstrauss theorem] Every conditionally weakly compactly generated Banach space has a conditionally weakly-$*$ sequentially compact dual unit ball.*

Finally we have a version of the perturbed James' compactness theorem (see [68, Theorem A.1], [86, Theorem 2], and [87]).

The next results follows by applying the transfer principle to its most general form (see [103, Theorem 2.4] and [84])

Theorem 3.3.8. *[Conditional unbounded version of James' Theorem] Let* $(\boldsymbol{E}, \|\cdot\|)$ *be a conditional Banach space and let* $\boldsymbol{f} : \boldsymbol{E} \to \overline{\boldsymbol{R}}$ *be a conditionally proper function such that*

$$\text{for all } \boldsymbol{x}^* \in \boldsymbol{E}^*, \text{ there is a } \boldsymbol{x}_0 \in \boldsymbol{E} \text{ so that } \boldsymbol{x}^*(\boldsymbol{x}_0) - \boldsymbol{f}(\boldsymbol{x}_0) = \sup_{\boldsymbol{x}\in\boldsymbol{E}} (\boldsymbol{x}^*(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{x})),$$

then for every conditional real number $\boldsymbol{r} \in \boldsymbol{R}$ *such that* $V_r(f) \neq \emptyset$, *the conditional* $\boldsymbol{r}$*-sub-level set* $\boldsymbol{V_r(f)}$ *is conditionally weakly relatively compact.*

The theorems above are just examples of application and the number of instances can be easily increased. In general, the different conditional versions of classical theorems proven in [32] are nothing else than reformulations of theorems of the form $[\![\varphi]\!] = I$ for some ZFC theorem φ. Thus all these theorems follow automatically from the transfer principle of Boolean-valued models and it is not needed a proof by conventional means as needed in [32].

Comments 3.3.1. *In the early stages of the research period preceding this work, we proved some of the theorems above by means of step-by-step proofs with the techniques of conditional set theory. The reader can find in Appendix B proofs for theorems 3.3.4, 3.3.5, 3.3.6, 3.3.7, and 3.3.8.*

Regarding Theorem 3.3.8, the statement proven in Section B.5 in the Appendix B is slightly weaker as we assume a technical assumption which simplifies the proof (see Theorem B.5.1).

3.4 Conditionally locally convex spaces and locally stable L^0-convex modules

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{A} := \bar{\mathcal{F}}$ the probability algebra associated to $\mathcal{F}$. In what follows, we will study the relations between the category of locally stable L^0-convex modules and L^0-module homomorphisms and the category of conditional locally convex vector spaces and conditionally linear continuous functions. We will see that both categories are in fact equivalent and, moreover, the underlying constructions will exhibit that the Boolean-valued principle established in Chapter 2 from convex analysis to L^0-analysis is a consequence of the Boolean-valued transfer principle derived in the present chapter.

When the underlying complete Boolean algebra is the measure algebra $\mathcal{A}$, we have that L^0 is isomorphic to $\mathbb{R}^{(\mathcal{A})}\!\downarrow$. The same happens for the generating set R of the conditional real numbers $\mathbf{R}$ associated to $\mathcal{A}$. In fact, in [32] it is shown that there is a conditional bijection between $\mathbf{R}$ and $\mathbf{L}^0$, which preserves the conditional order, conditional operations and the conditional absolute value. Therefore, in the following, we will identify R and L^0.

Proposition 3.4.1. *Let* $\boldsymbol{E}$ *be a conditional vector space. Consider the equivalence relation on* $E \times \mathcal{A}$: $(x,a) \sim (y,b)$ *if* $a = b$ *and* $1_a x = 1_b y$. *Then the quotient set* $\boldsymbol{E}'$ *is a conditional set of* E *such that the identity* $\mathrm{id} : E \to E$ *is a stable function when considering* E *in the left side as the generating set of* $\boldsymbol{E}$ *and in the right side as the generating set of* $\boldsymbol{E}'$. *In fact,* $\boldsymbol{E}'$ *is a conditional vector space and* $\mathbf{id} : \boldsymbol{E} \to \boldsymbol{E}'$ *is a conditional isomorphism of conditional vector spaces. In addition,* E *has the countable concatenation property and for any sequence* $(x_k) \subset E$ *and partition* $(a_k) \in p(I)$ *it holds* $\sum_k 1_{a_k} x_k = \sum x_k|a_k = \sum x_k|' a_k$, *where* $|$ *is the conditioning of* $\boldsymbol{E}$ *and* $|'$ *is the conditioning of* $\boldsymbol{E}'$.

Proof. The assertions of the statement follow easily if we prove the following

$$x|a = y|b \quad \text{if and only if} \quad a = b,\ 1_a x = 1_b y.$$

For any $a \in \mathcal{A}$ it is satisfied that $1_a = 1|a + 0|a^c$ in $\mathbf{R}$. Fix $x, y \in E$ and $a, b \in \mathcal{A}$. Then, by using the stability of the scalar product, one has

$$1_a x = (1|a + 0|a) \cdot (x|a + x|a^c) = 1 \cdot x|a + 0 \cdot x|a^c = x|a + 0|a^c. \tag{3.4}$$

Similarly,

$$1_b y = y|b + 0|b^c. \tag{3.5}$$

Let assume that $x|a = y|b$. By (C1), one has that $a = b$. Then, (3.4) and (3.5) yield $1_a x = 1_b y$.

Conversely, suppose that $1_a x = 1_b y$ and $a = b$. Again by (3.4) and (3.5), one has

$$x|a + 0|a^c = 1_a x = 1_a y = y|a + 0|a^c.$$

By (C2) we obtain that $x|a = y|a$ and, since $a = b$, we obtain $x|a = y|b$. □

Next, by means of a equivalence of categories, we show the connection between locally stable L^0-convex modules and conditionally locally convex spaces. Before we need two preliminary results:

Lemma 3.4.1. *[32, Proposition 3.5] Let $\boldsymbol{X}$ be a conditional set, $\mathscr{B}$ a stable collection of stable subsets of X and $\boldsymbol{\mathcal{B}}$ the conditional collection of conditional subsets of $\boldsymbol{X}$ generated by $\mathscr{B}$. Then $\boldsymbol{\mathcal{B}}$ is a conditional topological base on $\boldsymbol{X}$ if, and only if, $\mathscr{B}$ is a classical topological base on X. Moreover, if $\langle\mathscr{B}\rangle$ is the topology generated by $\mathscr{B}$ and $\langle\boldsymbol{\mathcal{B}}\rangle$ the conditional topology generated by $\boldsymbol{\mathcal{B}}$, then it holds*

$$\{O \in \langle\mathscr{B}\rangle \colon O \in \mathfrak{S}(\boldsymbol{X})\} = \{O \in \mathfrak{S}(\boldsymbol{X}) \colon \boldsymbol{O} \in \langle\boldsymbol{\mathcal{B}}\rangle\}.$$

Lemma 3.4.2. *[32, Proposition 3.11] Let $\boldsymbol{X}$ and $\boldsymbol{X}'$ be two conditional sets, $\mathscr{B}$ and $\mathscr{B}'$ stable collection of stable sets which are a base of a topology on X and X', respectively, and $\boldsymbol{\mathcal{B}}$ and $\boldsymbol{\mathcal{B}}'$ the conditional bases on $\boldsymbol{X}$ and $\boldsymbol{X}'$ generated by $\mathscr{B}$ and $\mathscr{B}'$, respectively. A conditional function $\boldsymbol{f} \colon \boldsymbol{X} \to \boldsymbol{X}'$ is conditionally continuous with respect to the conditional topologies $\langle\boldsymbol{\mathcal{B}}\rangle$ and $\langle\boldsymbol{\mathcal{B}}'\rangle$ if, and only if, $f : X \to X'$ is continuous with respect to the topologies $\langle\mathscr{B}\rangle$ and $\langle\mathscr{B}'\rangle$.*

The following result connects locally stable L^0-convex modules and conditionally convex spaces:

Theorem 3.4.1. *For fixed a probability space $(\Omega, \mathcal{F}, P)$ and the corresponding probability algebra $\mathcal{A} := \bar{\mathcal{F}}$, the following categories are equivalent:*

1. *The category of locally stable L^0-convex modules and continuous L^0-module homomorphisms;*
2. *The category of conditionally convex spaces and conditionally linear continuous function;*
3. *The category of names for locally convex spaces and linear continuous functions.*

Proof. The equivalence between the categories 1 and 3 is precisely Theorem 2.2.2.

Let us prove the equivalence between 1 and 2.

If $(E, \mathscr{T})$ is a locally stable L^0-convex module. Then we define $\mathbf{E}$ to be the quotient set of the relation of equivalence on $E \times \mathcal{A}$ given by

$$(x, a) \sim (y, b) \quad \text{whenever } a = b,\ 1_a x = 1_b y.$$

Let $x|a$ denote the equivalence class of (x, a). Then $\mathbf{E}$ is a conditional vector space. Let $\mathscr{U}$ be a stable neighborhood of $0 \in E$ consisting of stable L^0-convex subsets. Due to Lemma 3.4.1, the corresponding conditional set $\boldsymbol{\mathcal{U}}$ is a conditional neighborhood of $\mathbf{0} \in \mathbf{E}$ for a conditional locally convex topology on $\mathbf{E}$.

If $f : E \to E'$ is a continuous L^0-module homomorphism, the due to Lemma 3.4.2, $\mathbf{f} \colon \mathbf{E} \to \mathbf{E}'$ is a conditionally continuous linear function.

We define the functor $\Phi(E) := \mathbf{E}$, $\Phi(f) := \mathbf{f}$.

Conversely, if $\mathbf{E}$ is a conditionally locally convex space, then, due to Proposition 3.4.1, E is an L^0-module. If $\boldsymbol{\mathcal{U}}$ is a conditional neighborhood base of $\mathbf{0} \in \mathbf{E}$ whose conditional elements are conditionally convex, then $\mathscr{U} := \{U\colon \mathbf{U} \in \boldsymbol{\mathcal{U}}\}$ is a stable collection of stable subsets of E which is a neighborhood base of $0 \in E$ for some locally stable L^0-convex topology due to Lemma 3.4.1.

If $\mathbf{f} : \mathbf{E} \to \mathbf{E}'$ is conditionally linear continuous, then $f : E \to E'$ is a continuous L^0-module homomorphism due to Lemma 3.4.2. $\square$

Proposition 3.4.2. *Let $(E, \mathscr{T})$ be a locally stable L^0-convex module and let $(\boldsymbol{E}, \boldsymbol{\mathcal{T}}) := \Phi((E, \mathscr{T}))$, then:*

1. $\boldsymbol{\mathcal{T}} := \{\boldsymbol{O}|a \colon a \in \mathcal{A},\, O \in \mathscr{T} \cap \mathfrak{S}(E)\}$ *is a conditional topology on* $\boldsymbol{S}$*;*
2. $O \subset E$ *is stable and open if and only if* $\boldsymbol{O}$ *is conditionally open;*
3. $C \subset E$ *is stable and closed if and only if* $\boldsymbol{C}$ *is conditionally closed;*
4. $K \subset E$ *is stably compact if and only if* $\boldsymbol{K}$ *is conditionally compact;*

Proof. 1 and 2 are consequences of Lemma 3.4.1.

Let us prove 3. Let $\mathscr{U}$ be a stable neighborhood base of $0 \in E$ consisting of stable subsets of E, and consider the corresponding conditional collection $\boldsymbol{\mathcal{U}}$, which is a conditional neighborhood of $\mathbf{0} \in \mathbf{E}$.

$$\begin{aligned} C \text{ is closed } & \text{ if and only if } C = \bigcap_{U \in \mathscr{U}} (C + U) \\ & \text{ if and only if } \mathbf{C} = \bigsqcap_{\mathbf{U} \in \boldsymbol{\mathcal{U}}} (\mathbf{C} + \mathbf{U}) \\ & \text{ if and only if } \mathbf{C} \text{ is conditionally closed.} \end{aligned}$$

Let $\mathscr{G}$ be a stable collection of stable subsets of E and $\boldsymbol{\mathcal{G}}$ the corresponding conditional collection. Clearly, a stable collection $\mathscr{G}$ of stable subsets of E is a stable filter base if and only if $\boldsymbol{\mathcal{G}}$ is a conditional filter base. In that case, x is a cluster point of $\mathscr{G}$ if and only if $\mathbf{x}$ is a conditional cluster point of $\boldsymbol{\mathcal{G}}$. Then 3 follows. $\square$

Chapter 4

A Boolean-valued model approach to duality theory of conditional risk measures

This Chapter is aimed to establish a Boolean-valued transfer principle from duality theory of conventional risk measures to duality theory of conditional risk measures. Namely, we will provide a method that allows us to interpret any dual representation statement on conditional risk measures as a dual representation statement on conventional risk. Due to the Boolean-valued transfer principle, if the latter is a theorem, then the former is also a theorem.

In Section 4.1, we consider a probability space $(\Omega, \mathcal{E}, P)$ and a sub-σ-algebra $\mathcal{F} \subset \mathcal{E}$. Suppose that $\mathcal{A} := \bar{\mathcal{F}}$ is the probability algebra associated to $\mathcal{F}$, we show that we can find a tailored name $(\mathbb{\Omega}, \mathbb{\Sigma}, \mathbb{P})_{\mathcal{A}}$ for a probability space in such a way that $L^0(\mathcal{E})$ can be viewed as the space of $\mathbb{\Sigma}$-measurable random variables within $V^{(\mathcal{A})}$. Moreover, we see that the L^p type modules $L^p_{\mathcal{F}}(\mathcal{E})$ $(1 \leq p \leq \infty)$ can be interpreted as L^p spaces of $\mathbb{\Sigma}$-measurable random variables within $V^{(\mathcal{A})}$. We also recall the Orlicz type modules and introduce the Orlicz-heart type modules, and prove that they can be respectively interpreted as Orlicz and Orlicz-heart spaces of $\mathbb{\Sigma}$-measurable random variables within $V^{(\mathcal{A})}$. All this amounts to a Boolean-valued transfer principle; any available result on L^p, Orlicz, Orlicz-Heart spaces can respectively be interpreted as a theorem on L^p, Orlicz, or Orlicz-heart type modules. As an application, we prove a module analogue of the Kreps-Yan theorem.

In Section 4.2, by using the connections provided in Section 4.2, we study the relations between the different elements of the duality theory of conventional risk measures and the corresponding analogues of the duality theory of conditional risk measures. This finally gives rise to a Boolean-valued transfer principle from duality theory of conventional risk measures to duality theory of conditional risk measures.

In Section 4.3, we apply the transfer principle to establish a general robust representation of conditional risk measures, which follows from the corresponding theorem of conventional risk measures. We study the form of this result in different cases: the L^∞ type module, L^p type modules (with $1 \leq p < \infty$), Orlicz type modules, and Orlicz-heart type modules.

4.1 $L^0(\mathcal{F})$-modules of L^p, Orlicz, and Orlicz-heart type

Throughout, we will suppose that $(\Omega, \mathcal{E}, P)$ is a probability space such that $\mathcal{F} \subset \mathcal{E}$ is a sub-σ-algebra.

In Example 1.2.1 we recalled the L^p type modules $L^p_{\mathcal{F}}(\mathcal{E})$, which are module analogues of the classical L^p spaces of random variables. L^p type modules were proposed as a model space for conditional risk measure in [37]. We can define module analogues of other well-known spaces of measurable functions.

Recall that a *Young function* is a function $\phi : [0,\infty) \to [0,\infty]$ which is increasing, left-continuous, convex, finite on a neighborhood of 0 and such that $\varphi(0) = 0$ and $\lim_{x\to\infty} \phi(x) = \infty$.

Given a Young function ϕ, the associated Orlicz space is

$$L^\phi(\mathcal{E}) = \{x \in L^0(\mathcal{E})\colon \exists r \in (0,\infty),\ E_P[\phi(r^{-1}|x|)] < \infty\},$$

and the associated Orlicz-Heart space is

$$H^\phi(\mathcal{E}) = \{x \in L^0(\mathcal{E})\colon \forall r \in (0,\infty),\ E_P[\phi(r^{-1}|x|)] < \infty\}.$$

The function

$$\|\cdot\|_\phi : L^\phi(\mathcal{E}) \longrightarrow \mathbb{R}, \quad \|x\|_\phi = \inf\{r \in (0,\infty)\colon E_P[\phi(r^{-1}|x|)] < 1\}$$

is a norm which is called the *Luxemburg norm.* Both Orlicz and Orlicz-heart spaces endowed with the Luxemburg norm are Banach spaces.

One can define module analogues of Orlicz and Orlicz-heart spaces. The *Orlicz type modules* were introduced in [110]. We shall also introduce the *Orlicz-heart type modules*.

- We recall the Orlicz type module associated to ϕ, which is

$$L^\phi_{\mathcal{F}}(\mathcal{E}) := \{x \in L^0(\mathcal{E})\colon \exists \varepsilon \in L^0_{++}(\mathcal{F}),\ E_P[\phi(\varepsilon^{-1}|x|)|\mathcal{F}] \in L^0(\mathcal{F})\}.$$

  Then $L^\phi_{\mathcal{F}}(\mathcal{E})$ is the Orlicz type module associated to ϕ.

  The *Luxemburg $L^0(\mathcal{F})$-norm* $\|\cdot|\mathcal{F}\|_\phi : L^\phi_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ is

$$\|x|\mathcal{F}\|_\phi := \inf\{\varepsilon \in L^0(\mathcal{F})\colon E_P[\phi(\varepsilon^{-1}|x|)|\mathcal{F}] \le 1\} \quad \text{for } x \in L^\phi_{\mathcal{F}}(\mathcal{E}).$$

- The Orlicz-heart type module associated to ϕ is defined by

$$H^\phi_{\mathcal{F}}(\mathcal{E}) := \{x \in L^0(\mathcal{E})\colon \forall \varepsilon \in L^0_{++}(\mathcal{F}),\ E_P[\phi(\varepsilon^{-1}|x|)|\mathcal{F}] \in L^0(\mathcal{F})\}.$$

  Notice that $H^\phi_{\mathcal{F}}(\mathcal{E}) \subset L^\phi_{\mathcal{F}}(\mathcal{E})$. Thus $\|\cdot|\mathcal{F}\|_\phi$ also defines an $L^0(\mathcal{F})$-norm on $H^\phi_{\mathcal{F}}(\mathcal{E})$.

In this section, we will prove that the L^p, Orlicz and Orlicz-Heart type modules can respectively be viewed as descents of names for L^p, Orlicz and Orlicz-heart spaces of random variables of some probability space within the model $V^{(\mathcal{A})}$, where $\mathcal{A} := \bar{\mathcal{F}}$ is the probability algebra associated to $\mathcal{F}$.

This means that any known theorem on L^p, Orlicz or Orlicz-heart spaces of random variables has an interpretation which gives rises to a new theorem on L^p, Orlicz or Orlicz-heart type modules, respectively, thanks to the Boolean-valued transfer principle.

Obviously, $L^0(\mathcal{E})$ is an $L^0(\mathcal{F})$-module, therefore we can consider its ascent as in Chapter 2. In what follows, we will construct a more elaborated ascent $L^0(\mathcal{E})\!\uparrow$ so that there exists a name $(\mathbb{\Omega},\mathbb{\Sigma},\mathbb{P})_{\mathcal{A}}$ for a probability space with $[\![L^0(\mathcal{E})\!\uparrow = L^0(\mathbb{\Sigma})]\!] = I$. The construction needs some steps.

Let $\bar{\mathcal{E}}$ denote the probability algebra associated to $\mathcal{E}$. We can suppose that $\mathcal{A}$ is a Boolean subalgebra of $\bar{\mathcal{E}}$. Next, we will define a suitable ascent of $\bar{\mathcal{E}}$.

- For any $e \in \bar{\mathcal{E}}$ we define $\tilde{e}$ to be the representative in $\overline{V}^{(\mathcal{A})}$ of the name given by the function
$$\{\check{f} \colon f \in \bar{\mathcal{E}}\} \longrightarrow \mathcal{A} \quad \check{f} \mapsto a_{e,f},$$
where $a_{e,f} := \bigvee\{a \in \mathcal{A} \colon a \wedge e = a \wedge f\}$.

- Let $\bar{\mathcal{E}}\!\uparrow$ denote the representative in $\overline{V}^{(\mathcal{A})}$ of the name given by the function
$$\{\tilde{e} \colon e \in \bar{\mathcal{E}}\} \longrightarrow \mathcal{A} \quad \tilde{e} \mapsto I.$$

Consider the following equivalence relation on $\bar{\mathcal{E}} \times \mathcal{A}$ given by

$$(e,a) \sim (f,b) \quad \text{if and only if } a = b,\ e \wedge a = f \wedge a.$$

Denote by $\bar{\boldsymbol{\mathcal{E}}}$ the quotient set and by

$$\bar{\mathcal{E}} \times \mathcal{A} \longrightarrow \bar{\boldsymbol{\mathcal{E}}} \quad (e,a) \mapsto e|a$$

the projection. Then it follows by inspection that $\bar{\boldsymbol{\mathcal{E}}}$ is a conditional set of $\bar{\mathcal{E}}$ with $\sum e_k|a_k = \bigvee (e_k \wedge a_k)$ for all $(a_k) \in p(I)$ and $(e_k) \subset \bar{\mathcal{E}}$. Besides, by construction, note that $\bar{\boldsymbol{\mathcal{E}}}\!\uparrow$ is precisely $\bar{\mathcal{E}}\!\uparrow$.

Thus, the following is immediate from Proposition 3.1.7.

Proposition 4.1.1. *The map $i_{\mathcal{E}} : \bar{\mathcal{E}} \to \bar{\mathcal{E}}\!\uparrow\downarrow$, $e \mapsto \tilde{e}$ is a bijection such that for any partition $(a_k) \in p(I)$ and countable family $(e_k) \subset \bar{\mathcal{E}}$ it is satisfied that $i_{\mathcal{E}}(\bigvee_k (a_k \wedge e_k)) = \sum_k i_{\mathcal{E}}(e_k) a_k$.*

Likewise, the following is a consequence of Lemma 3.1.1:

Lemma 4.1.1. *For any $e, f \in \bar{\mathcal{E}}$ one has that $[\![\tilde{e} = \tilde{f}]\!] = a_{e,f}$.*

Lemma 4.1.2. *If $e, f, g, h \in \bar{\mathcal{E}}$, then:*

1. $a_{e,g} \wedge a_{f,h} \leq a_{e \vee f, g \vee h}$;
2. $a_{e,g} \wedge a_{f,h} \leq a_{e \wedge f, g \wedge h}$;
3. $a_{e,f} = a_{e^c, f^c}$.

Proof. Set $a := a_{e,g} \wedge a_{f,h}$. If follows that $e \wedge a = g \wedge a$ and $f \wedge a = g \wedge h$. Then

$$a \wedge (e \vee f) = (a \wedge e) \vee (a \wedge f) = (a \wedge g) \vee (a \wedge h) = a \wedge (g \vee h).$$

This proves 1. A similar argument proves 2. Let us prove 3. Given $a \in \mathcal{A}$, we have

$$\begin{aligned} a \wedge e^c = a \wedge f^c \quad & \text{if and only if} \quad a^c \vee e = a^c \vee f \\ & \text{if and only if} \quad a \wedge (a^c \vee e) = a \wedge (a^c \vee f) \\ & \text{if and only if} \quad a \wedge e = a \vee f. \end{aligned}$$

Then 3 follows. $\square$

Proposition 4.1.2. *$\bar{\mathcal{E}}\!\uparrow$ is a name for a complete Boolean algebra. Moreover, for any $e, f \in \bar{\mathcal{E}}$, one has $[\![(e \vee f)^\sim = \tilde{e} \vee \tilde{f}]\!] = I$, $[\![(e \wedge f)^\sim = \tilde{e} \wedge \tilde{f}]\!] = I$, $[\![(e^c)^\sim = \tilde{e}^c]\!] = I$, $[\![\tilde{0} = 0]\!] = I$ and $[\![\tilde{I} = I]\!] = I$. In addition, $\bar{\mathcal{E}}\!\uparrow\downarrow$ is a complete Boolean algebra and $i_{\mathcal{E}}$ is an isomorphism of Boolean algebras between $\bar{\mathcal{E}}$ and $\bar{\mathcal{E}}\!\uparrow\downarrow$.*

Proof. We consider the functions

$$\lambda_\vee : \bar{\mathcal{E}}\!\uparrow\downarrow \times \bar{\mathcal{E}}\!\uparrow\downarrow \to \bar{\mathcal{E}}\!\uparrow\downarrow \quad \lambda_\vee(\tilde{e}, \tilde{f}) := (e \vee f)^\sim;$$

$$\lambda_\wedge : \bar{\mathcal{E}}\!\uparrow\downarrow \times \bar{\mathcal{E}}\!\uparrow\downarrow \to \bar{\mathcal{E}}\!\uparrow\downarrow \quad \lambda_\wedge(\tilde{e}, \tilde{f}) := (e \wedge f)^\sim;$$

$$\lambda_c : \bar{\mathcal{E}}\!\uparrow\downarrow \to \bar{\mathcal{E}}\!\uparrow\downarrow \quad \lambda_c(\tilde{e}) := (e^c)^\sim.$$

Lemmas 4.1.1 and 4.1.2 prove that these functions are extensional. Theorems 2.1.5 and 2.1.6 provide us with names $\tilde{\lambda}_\vee$, $\tilde{\lambda}_\wedge$ for functions from $(\bar{\mathcal{E}}\!\uparrow \times \bar{\mathcal{E}}\!\uparrow)_{\mathcal{A}}$ to $\bar{\mathcal{E}}\!\uparrow$ and a name $\tilde{\lambda}_c$ for a function from $\bar{\mathcal{E}}\!\uparrow$ to $\bar{\mathcal{E}}\!\uparrow$, so that

$$[\![\tilde{\lambda}_\vee(\tilde{e}, \tilde{f}) = (e \vee f)^\sim]\!] = I, \quad [\![\tilde{\lambda}_\wedge(\tilde{e}, \tilde{f}) = (e \wedge f)^\sim]\!] = I \quad \text{for all } e, f \in \bar{\mathcal{E}},$$

and

$$[\![\tilde{\lambda}_c(\tilde{e}) = (e^c)^\sim]\!] = I \quad \text{for all } e \in \bar{\mathcal{E}}.$$

Clearly, $(\bar{\mathcal{E}}\!\uparrow, \tilde{\lambda}_\vee, \tilde{\lambda}_\wedge, \tilde{\lambda}_c, \tilde{I}, \tilde{0})_{\mathcal{A}}$ is a name for a Boolean algebra. Besides, since $\bar{\mathcal{E}}$ is complete, my means of Theorem 2.1.4 it can be verified that $\bar{\mathcal{E}}\!\uparrow$ is a name for a complete Boolean algebra.

Due to the maximum principle, $\bar{\mathcal{E}}\!\uparrow\downarrow$ inherits the Boolean operations of $\bar{\mathcal{E}}\!\uparrow$. Since $i_{\mathcal{E}}$ is a bijection which preserves the Boolean operations, it is an isomorphism of Boolean algebras. □

For $e \in \bar{\mathcal{E}}$, we define the *conditional probability* of e by $\bar{P}(e|\mathcal{F}) := E_P[1_e|\mathcal{F}]$.

Since $e \mapsto \tilde{e}$ is a bijection the map

$$\bar{\mathcal{E}}\!\uparrow\downarrow \longrightarrow \mathbb{R}_{\mathcal{A}}\!\downarrow, \quad \tilde{e} \mapsto \bar{P}(e|\mathcal{F})$$

is well-defined. This function is also extensional, thus Theorem 2.1.5 yields a name $\bar{P}\!\uparrow$ for a function from $\bar{\mathcal{E}}\!\uparrow$ to $\mathbb{R}_{\mathcal{A}}$ such that

$$[\![\bar{P}(e|\mathcal{F})^\bullet = \bar{P}\!\uparrow(\tilde{e})]\!] = I \quad \text{for all } e \in \bar{\mathcal{E}}.$$

An elementary manipulation of the Boolean truth values proves the following:

Proposition 4.1.3. *$(\bar{\mathcal{E}}\!\uparrow, \bar{P}\!\uparrow)_{\mathcal{A}}$ is a name for a probability algebra.*

The proposition above asserts that $(\bar{\mathcal{E}}\!\uparrow, \bar{P}\!\uparrow)_{\mathcal{A}}$ is a name for a probability algebra. However, since Ω is not used in the construction of $\bar{\mathcal{E}}\!\uparrow$, it is not clear if $(\bar{\mathcal{E}}\!\uparrow, \bar{P}\!\uparrow)_{\mathcal{A}}$ is a name for the probability algebra associated to a probability space. Since our main purpose is to define a name for a space of measure functions, we need a name for a probability space whose associated probability algebra is precisely $\bar{\mathcal{E}}\!\uparrow$.

This problem is solved by the following variation of the representation Stone theorem:

Proposition 4.1.4. *[42, 321J] Any probability algebra $(\mathcal{B}, \mu)$ is isomorphic, as probability algebra, to the probability algebra $(\bar{\Sigma}, \bar{Q})$ associated to some probability space (X, Σ, Q).*

The transfer and maximum principles applied to the proposition above yields the following:

Proposition 4.1.5. *There exists a name $(\mathbb{\Omega}, \mathbb{\Sigma}, \mathbb{P})_{\mathcal{A}}$ for a probability space such that, if $(\bar{\mathbb{\Sigma}}, \bar{\mathbb{P}})_{\mathcal{A}}$ is a name for the measure algebra associated to $(\mathbb{\Omega}, \mathbb{\Sigma}, \mathbb{P})_{\mathcal{A}}$, then there exists a name for an isomorphism of probability algebras between $(\bar{\mathcal{E}}\!\uparrow, \bar{P}\!\uparrow)_{\mathcal{A}}$ and $(\bar{\mathbb{\Sigma}}, \bar{\mathbb{P}})_{\mathcal{A}}$.*

Given the result above we can already define the ascent of $L^0(\mathcal{E})$. For the forthcoming discussion, we will fix the canonical name $(\mathbb{\Omega}, \mathbb{\Sigma}, \mathbb{P})_{\mathcal{A}}$ for a probability space as in the proposition above. Consider the canonical name $L^0(\mathbb{\Sigma})_{\mathcal{A}}$ for the space of (equivalence classes of) $\mathbb{\Sigma}$-measurable random measures. We define $L^0(\mathcal{E})\!\uparrow := L^0(\mathbb{\Sigma})_{\mathcal{A}}$.

Since $\bar{\mathcal{E}}\!\uparrow$ and $\bar{\mathbb{\Sigma}}_{\mathcal{A}}$ are isomorphic Boolean algebras within the model $\overline{V}^{(\mathcal{A})}$, we can find an isomorphism of Boolean algebras:

$$\begin{array}{ccc} \bar{\mathcal{E}} & \longrightarrow & \bar{\mathbb{\Sigma}}_{\mathcal{A}}\!\downarrow \\ e & \longmapsto & e^\bullet \\ \mathbb{e}^\circ & \longleftarrow & \mathbb{e} \end{array}$$

Definition 4.1.1. *Say that a family $(e_\eta)_{\eta\in L^0(\mathcal{F})}$ in $\bar{\mathcal{E}}$ has the* local property *if*

$$a \wedge e_{1_a\eta} = a \wedge e_\eta \quad \textit{for all } \eta \in L^0(\mathcal{F}),\ a \in \mathcal{A}.$$

Note that if $(e_\eta)_{\eta\in L^0(\mathcal{F})}$ has the local property, then the mapping

$$\mathbb{R}_{\mathcal{A}}\!\downarrow \longrightarrow \bar{\mathcal{E}}\!\uparrow\downarrow \quad \eta^\bullet \mapsto e^\bullet_\eta$$

is extensional. Due to Theorem 2.1.5 we can define the corresponding ascent $(e_\eta)\!\uparrow$. Then the following holds:

Lemma 4.1.3. *Let $(e_\eta)_{\eta\in L^0}$ be family of elements of $\bar{\mathcal{E}}$ with the local property. It is satisfied:*

1. $[\![\bigvee_{r\in\mathbb{R}} e^\bullet_{r^\circ} = (\bigvee_{\eta\in L^0(\mathcal{F})} e_\eta)^\bullet]\!] = I$;
2. $[\![\bigwedge_{r\in\mathbb{R}} e^\bullet_{r^\circ} = (\bigwedge_{\eta\in L^0(\mathcal{F})} e_\eta)^\bullet]\!] = I$.

Proof. The assertions follow from the fact that $e \mapsto e^\bullet$ is an isomorphism of Boolean algebras together with Theorem 2.1.4. $\square$

For given $x \in L^0(\mathcal{E})$, we have that

$$\bigvee_{\eta\in L^0(\mathcal{F})} \{x \geq \eta\} = I \quad \text{and} \quad \bigwedge_{\eta\in L^0(\mathcal{F})} \{x \geq \eta\} = 0,$$

and for any $\xi \in L^0(\mathcal{F})$ we have that

$$\{x \geq \xi\} = \bigvee_{\eta\geq\xi} \{x \geq \eta\}.$$

In virtue of Lemma 4.1.3, we have:

$$\left[\!\!\left[\bigvee_{r\in\mathbb{R}} \{x \geq r^\circ\}^\bullet = I\right]\!\!\right] = I, \quad \left[\!\!\left[\bigwedge_{r\in\mathbb{R}} \{x \geq r^\circ\}^\bullet = 0\right]\!\!\right] = I, \tag{4.1}$$

and

$$\left[\!\!\left[(\forall s \in \mathbb{R}) \left(\{x \geq s^\circ\}^\bullet = \bigvee_{r\geq s} \{x \geq r^\circ\}^\bullet\right)\right]\!\!\right] = I. \tag{4.2}$$

The proofs of the following two lemmas are elementary exercises:

Lemma 4.1.4. *Suppose that (X, Σ, Q) is a probability space. Let $(a_r)_{r\in\mathbb{R}}$ be a family in $\bar{\Sigma}$ such that:*

1. $\bigvee_{r\in\mathbb{R}} a_r = I$,
2. $\bigwedge_{r\in\mathbb{R}} a_r = 0$,
3. $a_s = \bigvee_{r\geq s} a_s$ *for all* $s \in \mathbb{R}$.

Then

$$x := \sup_{r\in\mathbb{R}}\{r1_{a_r} - \infty 1_{a_r^c}\} \tag{4.3}$$

is in $L^0(\Sigma)$ *and* $\{x \geq r\} = a_r$ *for all* r. *Conversely, any* $x \in L^0(\Sigma)$ *admits a representation (4.3) with* $a_r := \{x \geq r\}$ *for all* $r \in \mathbb{R}$.

Lemma 4.1.5. *Suppose that* (X, Σ, Q) *is a probability space. Then, for* $x, y \in L^0(\Sigma)$ *and* $r \in \mathbb{R}$*:*

1. $\{x + y \geq r\} = \bigvee_{s\in\mathbb{R}}\{x \geq s\} \wedge \{y \geq r - s\}$;
2. $\{xy \geq r\} = \bigvee_{s\in\mathbb{R},\, s>0}\{x \geq s\} \wedge \{y \geq \frac{r}{s}\}$;
3. $\{x \vee y \geq r\} = \{x \geq r\} \vee \{y \geq r\}$.

For any given $x \in L^0(\mathcal{E})$, the transfer principle applied to Lemma 4.1.4 together with conditions (4.1) and (4.2) yield the existence of a unique element $x^\bullet \in L^0(\mathbb{\Sigma})_{\mathcal{A}}{\downarrow}$ satisfying

$$\left[\!\!\left[x^\bullet = \sup_{r\in\mathbb{R}}\{r1_{\{x\geq r^\circ\}^\bullet} - \infty 1_{\{x<r^\circ\}^\bullet}\}\right]\!\!\right] = I \quad \text{and} \quad [\![\{x^\bullet \geq \eta^\bullet\} = \{x \geq \eta\}^\bullet]\!] = I \text{ for all } \eta \in L^0(\mathcal{F}).$$

Proposition 4.1.6. *The correspondence* $x \mapsto x^\bullet$ *is an isomorphism of lattice ordered rings from* $L^0(\mathcal{E})$ *to* $L^0(\mathcal{E}){\uparrow}{\downarrow}$ *which extends the canonical isomorphism* $\eta \mapsto \eta^\bullet$ *between* $L^0(\mathcal{F})$ *and* $\mathbb{R}_{\mathcal{A}}{\downarrow}$ *(identifying* $\mathbb{R}_{\mathcal{A}}$ *with the constant functions inside* $V^{(\mathcal{A})}$*).*

Proof. By means of the maximum principle, $L^0(\mathbb{\Sigma})_{\mathcal{A}}{\downarrow}$ inherits the ring operations and order of $L^0(\mathbb{\Sigma})_{\mathcal{A}}$ and becomes a lattice ordered ring. Given $x \in L^0(\mathcal{E})$, by construction we have that

$$\{x \geq \eta\}^\bullet = \{x^\bullet \geq \eta^\bullet\}_{\mathcal{A}} \quad \text{for all } \eta \in L^0(\mathcal{F}). \tag{4.4}$$

In particular, if $x^\bullet = y^\bullet$, then we have that $\{x \geq \eta\} = \{y \geq \eta\}$ for all $\eta \in L^0(\mathcal{F})$ and thus, $x = y$. This proves the injectivity.

If $\mathbb{u} \in L^0(\mathbb{\Sigma})_{\mathcal{A}}{\downarrow}$, then $[\![\mathbb{u} = \sup_{r\in\mathbb{R}}\{r1_{\{\mathbb{u}\geq r\}} - \infty 1_{\{\mathbb{u}<r\}}\}]\!] = I$. We have that

$$x := \sup_{\eta\in L^0(\mathcal{F})}\{\eta 1_{\{\mathbb{u}\geq\eta^\bullet\}^\circ} - \infty 1_{\{\mathbb{u}<\eta^\bullet\}^\circ}\}$$

is a preimage of $\mathbb{u}$.

To prove that $(x + y)^\bullet = x^\bullet + y^\bullet$, it suffices to show that

$$[\![(\forall r \in \mathbb{R})\{(x + y)^\bullet \geq r\} = \{x^\bullet + y^\bullet \geq r\}]\!] = I.$$

The transfer principle applied to 1 of Lemma 4.1.5 shows that, for $\mathbb{r} \in \mathbb{R}_{\mathcal{A}}{\downarrow}$, one has

$$\left[\!\!\left[\{x^\bullet + y^\bullet \geq \mathbb{r}\} = \bigvee_{s\in\mathbb{R}}\{x^\bullet \geq \mathbb{r}\} \wedge \{y^\bullet \geq \mathbb{r} - s\}\right]\!\!\right] = I$$

Thus it suffices to show that for any $\eta \in L^0(\mathcal{F})$

$$\left[\!\!\left[\{(x + y)^\bullet \geq \eta^\bullet\} = \bigvee_{s\in\mathbb{R}}\{x^\bullet \geq \eta^\bullet\} \wedge \{y^\bullet \geq \eta^\bullet - s\}\right]\!\!\right] = I. \tag{4.5}$$

Now, we have that

$$
\begin{aligned}
(4.5) \quad & \text{if and only if} \quad \{(x+y)^\bullet \geq \eta^\bullet\}_{\mathcal{A}} = \bigvee_{\xi \in L^0(\mathcal{F})} \{x^\bullet \geq \eta^\bullet\}_{\mathcal{A}} \wedge \{y^\bullet \geq \eta^\bullet - \xi^\bullet\}_{\mathcal{A}} \text{ in } \bar{\Sigma}\!\downarrow \\
& \text{if and only if} \quad \{(x+y) \geq \eta\}^\bullet = \bigvee_{\xi \in L^0(\mathcal{F})} \{x \geq \eta\}^\bullet \wedge \{y \geq \eta - \xi\}^\bullet \text{ in } \bar{\Sigma}\!\downarrow \\
& \text{if and only if} \quad \{x+y \geq \eta\} = \bigvee_{\xi \in L^0(\mathcal{F})} \{x \geq \eta\} \wedge \{y \geq \eta - \xi\} \text{ in } \bar{\mathcal{E}},
\end{aligned}
$$

where in the second equivalence we have used (4.4) and in the third one we have used that $e \mapsto e^\bullet$ is an isomorphism of Boolean algebras. It is not difficult to show that the last condition holds. So the conclusion follows.

Similar arguments applied to 2 and 3 of Lemma 4.1.5 prove that $x^\bullet y^\bullet = (xy)^\bullet$ and $x^\bullet \vee y^\bullet = (x \vee y)^\bullet$, respectively.

□

Proposition above provides us with a stable isomorphism of lattice ordered rings

$$
\begin{array}{ccc}
L^0(\mathcal{E}) & \longrightarrow & L^0(\mathcal{E})\!\uparrow\downarrow \\
x & \longmapsto & x^\bullet \\
\mathbb{x}^\circ & \longleftarrow & \mathbb{x}.
\end{array}
$$

This justifies the definition $L^0(\mathcal{E})\!\uparrow := L^0(\Sigma)_{\mathcal{A}}$.

Remark 4.1.1. *We can prove that, as in Lemma 2.2.2, we have that*

$$
[\![x^\bullet = y^\bullet]\!] = a_{x,y} \quad \textit{for all } x, y \in L^0(\mathcal{E}). \tag{4.6}
$$

This means that $L^0(\mathcal{E})\!\uparrow$ has similar properties as the ascent of a general L^0-module constructed in Chapter 2. In fact, the only property we need to develop Chapter 2 is Lemma 2.2.2. Thus the properties studied in Chapter 2 are still valid for this new ascent operation.

Let us prove (4.6). Indeed, fixed $x, y \in L^0(\mathcal{E})$, let $a \in \mathcal{A}$. Then

$$
\begin{aligned}
1_a x = 1_a y \quad & \textit{if and only if} \quad [\![(1_a x)^\bullet = (1_a y)^\bullet]\!] = I \\
& \textit{if and only if} \quad [\![1_a^\bullet x^\bullet = 1_a^\bullet y^\bullet]\!] = I \\
& \textit{if and only if} \quad [\![(1_a^\bullet = 1) \wedge (x^\bullet = y^\bullet)]\!] \vee [\![1_a^\bullet = 0]\!] = I \\
& \textit{if and only if} \quad (a \wedge [\![x^\bullet = y^\bullet]\!]) \vee a^c = I \\
& \textit{if and only if} \quad a \leq [\![x^\bullet = y^\bullet]\!].
\end{aligned}
$$

Given a sequence (x_n) in $L^0(\mathcal{E})$ we define the stable sequence $(x_{\mathfrak{n}})_{\mathfrak{n} \in L^0(\mathcal{F},\mathbb{N})}$ with $x_{\mathfrak{n}} = \sum_k 1_{\{\mathfrak{n}=k\}} x_k$. Then the function

$$
\mathbb{N}_{\mathcal{A}}\!\downarrow \longrightarrow L^0(\Sigma)_{\mathcal{A}}\!\downarrow \quad \mathfrak{n}^\bullet \mapsto x_{\mathfrak{n}}^\bullet,
$$

is extensional and we can therefore consider the corresponding ascent $(\mathbb{x}_{\mathfrak{n}})\!\uparrow$ which is a name for a sequence in $L^0(\mathcal{E})\!\uparrow$.

Proposition 4.1.7. *If (x_n) is a sequence in $L^0(\mathcal{E})$, then*

$$
\lim_n x_n = x \text{ a.s. in } L^0(\mathcal{E}) \quad \text{if and only if} \quad [\![\lim_n x_{n^\circ}^\bullet = x^\bullet \text{ a.s. in } L^0(\mathcal{E})\!\uparrow]\!] = I.
$$

Proof. On one hand, we have that

$$[\![\limsup_n x^\bullet_{n^\circ} = (\inf_{\mathfrak{m}} \sup_{\mathfrak{n}\geq\mathfrak{m}} x_{\mathfrak{n}})^\bullet]\!] = I.$$

On the other, we have that $\inf_{\mathfrak{m}} \sup_{\mathfrak{n}\geq\mathfrak{m}} x_{\mathfrak{n}} = \inf_m \sup_{n\geq m} x_n = \limsup_n x_n$.

Thus $[\![\limsup_n x^\bullet_{n^\circ} = (\limsup_n x_n)^\bullet]\!] = I$.

Similarly, we have $[\![\liminf_n x^\bullet_{n^\circ} = (\liminf_n x_n)^\bullet]\!] = I$. Thus, we obtain the assertion. □

Now, we define $L^1_{\mathcal{F}}(\mathcal{E})\!\uparrow := L^1(\mathbb{\Sigma})_{\mathcal{A}}$, where $L^1(\mathbb{\Sigma})_{\mathcal{A}}$ is the interpretation in $\overline{V}^{(\mathcal{A})}$ of the space of (classes of equivalence) of $\mathbb{\Sigma}$-measurable random variables with finite expectation. The following proposition justifies this definition.

Proposition 4.1.8. *The mapping $x \mapsto x^\bullet$ bijectes $L^1_{\mathcal{F}}(\mathcal{E})$ into $L^1(\mathbb{\Sigma})_{\mathcal{A}}\!\downarrow$. Moreover,*

$$[\![E_P[x|\mathcal{F}]^\bullet = E_{\mathbb{P}}[x^\bullet]]\!] = I \quad \text{for all } x \in L^1_{\mathcal{F}}(\mathcal{E}).$$

Proof. Given $e \in \bar{\mathcal{E}}$, it is satisfied that

$$E_P[1_e|\mathcal{F}] = \bar{P}(e|\mathcal{F}), \quad [\![\bar{P}(e|\mathcal{F})^\bullet = \bar{\mathbb{P}}(e^\bullet)]\!] = I, \quad [\![\bar{\mathbb{P}}(e^\bullet) = E_{\mathbb{P}}[1_{e^\bullet}]]\!] = I \quad \text{and} \quad 1_{e^\bullet} = 1_e^\bullet.$$

Hence, we have that $[\![E_P[1_e|\mathcal{F}]^\bullet = E_{\mathbb{P}}[1_e{}^\bullet]]\!] = I$.

If $s \in L^0_+(\mathcal{E})$ is a simple function, by linearity it follows $[\![E_P[s|\mathcal{F}]^\bullet = E_{\mathbb{P}}[s^\bullet]]\!] = I$.

Now suppose that $x \in L^1_{\mathcal{F}}(\mathcal{E})$ with $x \geq 0$. We can find a sequence (s_n) of simple functions such that $0 \leq s_n \nearrow x$ a.s.. For each $\mathfrak{n} \in L^0(\mathcal{F},\mathbb{N})$, define $s_{\mathfrak{n}} = \sum_k 1_{\{\mathfrak{n}=k\}} s_k$. Proposition 4.1.7 proves that

$$[\![0 \leq s^\bullet_{n^\circ} \nearrow x^\bullet \text{ a.s.}]\!] = I.$$

By the transfer principle applied to the monotone convergence theorem we have that

$$[\![\lim_n E_{\mathbb{P}}[s^\bullet_{n^\circ}] = E_{\mathbb{P}}[x^\bullet]]\!] = I.$$

Let $\eta \in \bar{L}^0(\mathcal{F})$, $(\eta_n) \subset L^0(\mathcal{F})$ be with $[\![\eta^\bullet = E_{\mathbb{P}}[x^\bullet]]\!] = I$ and $[\![\eta^\bullet_n = E_{\mathbb{P}}[s^\bullet_{n^\circ}]]\!] = I$ for all $n \in \mathbb{N}$. Notice that $\eta_n = E_P[s_n|\mathcal{F}]$. Then, by Proposition 2.1.5, we have that $\lim_n E_P[s_n|\mathcal{F}] = \eta$ a.s., and by monotone convergence $\lim_n E_P[s_n|\mathcal{F}] = E_P[x|\mathcal{F}]$ a.s.. Thereby, we conclude that

$$[\![E_P[x|\mathcal{F}]^\bullet = E_{\mathbb{P}}[x^\bullet]]\!] = I.$$

Any $x \in L^1_{\mathcal{F}}$ is of the form $x = x^+ - x^-$. Thus, by linearity, the equality extends to any $x \in L^1_{\mathcal{F}}(\mathcal{E})$. This also proves that $x^\bullet \in L^1(\mathbb{\Sigma})_{\mathcal{A}}\!\downarrow$ whenever $x \in L^1_{\mathcal{F}}(\mathcal{E})$.

Now, if $\mathbb{u} \in L^1(\mathbb{\Sigma})_{\mathcal{A}}\!\downarrow$, take $x := \mathbb{u}^\circ \in L^0(\mathcal{E})$. Then, for any $\mathfrak{n} \in L^0(\mathcal{F},\mathbb{N})$, it follows that $|x| \wedge \mathfrak{n} \in L^1_{\mathcal{F}}(\mathcal{E})$. Therefore, we have $[\![E_P[|x| \wedge \mathfrak{n}|\mathcal{F}]^\bullet = E_{\mathbb{P}}[(|x| \wedge \mathfrak{n})^\circ]]\!] = I$. By the transfer principle applied to the monotone convergence theorem we obtain that

$$[\![\lim_n E_P[|x| \wedge n^\circ|\mathcal{F}]^\bullet = E_{\mathbb{P}}[|x|^\circ]]\!] = I.$$

By Proposition 2.1.5, we conclude that $\lim_n E_P[|x| \wedge n|\mathcal{F}]$ is finite and thus $x \in L^1_{\mathcal{F}}(\mathcal{E})$. □

If $1 < p \leq \infty$, we have that $\check{p}$ is a name for a real number with $[\![1 \leq \check{p} \leq \infty]\!] = I$. Consequently, we can consider the canonical name, say $L^p(\mathbb{\Sigma})_{\mathcal{A}}$, for the corresponding space of measurable functions. We define $L^p_{\mathcal{F}}(\mathcal{E})\!\uparrow := L^p(\mathbb{\Sigma})_{\mathcal{A}}$. Now suppose that $\phi : [0,\infty) \to [0,\infty]$ is a Young function. Consider the mapping $\eta \mapsto \phi(\eta)$ from $L^0([0,\infty))$ to $\bar{L}^0([0,\infty])$, which will be denoted again by ϕ. We can therefore take the ascent $\phi\!\uparrow$ which is a name for a function from $[0,\infty)_{\mathcal{A}}$ to $[0,\infty]_{\mathcal{A}}$. Moreover, $\phi\!\uparrow$ is a name for a Young function. Therefore, we can also consider canonical names, say $L^\phi(\mathbb{\Sigma})_{\mathcal{A}}$ and $H^\phi(\mathbb{\Sigma})_{\mathcal{A}}$, for the corresponding Orlicz and Orlicz-heart spaces, respectively. We define $L^\phi_{\mathcal{F}}(\mathcal{E})\!\uparrow := L^\phi(\mathbb{\Sigma})_{\mathcal{A}}$ and $H^\phi_{\mathcal{F}}(\mathcal{E})\!\uparrow := H^\phi(\mathbb{\Sigma})_{\mathcal{A}}$. The next result is an immediate consequence of Proposition 4.1.8 and justifies these definitions:

Corollary 4.1.1. *If $1 \leq p \leq \infty$, then $x \mapsto x^\bullet$ is a bijection between $L^p_{\mathcal{F}}(\mathcal{E})$ and $L^p_{\mathcal{F}}(\mathcal{E})\uparrow\downarrow$. If ϕ is a Young function, then $x \mapsto x^\bullet$ is a bijection between $L^\phi_{\mathcal{F}}(\mathcal{E})$ and $L^\phi_{\mathcal{F}}(\mathcal{E})\uparrow\downarrow$, and between $H^\phi_{\mathcal{F}}(\mathcal{E})$ and $H^\phi_{\mathcal{F}}(\mathcal{E})\uparrow\downarrow$.*

As explained in Remark 4.1.1, we can carry out a similar discussion as in Chapter 2: we can defined the ascent operations $\cdot\uparrow$ and $\cdot\Uparrow$ for stable subsets of $L^0(\mathcal{E})$ and stable collections of stable subsets of $L^0(\mathcal{E})$, respectively; and the descent operations $\cdot\downarrow$ and $\cdot\Downarrow$ for names of non-empty subsets of $L^0(\mathcal{E})\uparrow$ and names for non-empty collections of non-empty subsets of $L^0(\mathcal{E})\uparrow$, respectively.

This also applies to the $L^0(\mathcal{F})$-modules $L^p_{\mathcal{F}}(\mathcal{E})$, $L^\phi_{\mathcal{F}}(\mathcal{E})$ and $H^\phi_{\mathcal{F}}(\mathcal{E})$, which are bijected into descents of names of L^p, Orlicz and Orlicz-heart spaces, respectively, preserving operations, order and expectations, due to Proposition 4.1.8 and Corollary 4.1.1.

Thus, due to the transfer principle any theorem on L^p, Orlicz and Orlicz-heart spaces of random variables has a counterpart on L^p, Orlicz, and Orlicz-heart type modules, respectively. For instance, the Luxemburg $L^0(\mathcal{F})$-norm can be interpreted as the classical Luxemburg norm, and the Orlicz and Orlicz-Heart type modules are stably complete modules, which are interpretations of Banach spaces.

As an example of application, let us see the interpretation of a well-result on L^p space:

The Kreps-Yan theorem has a key role in the theory of non-arbitrage criteria (see eg [30, Theorem 5.2.3]). Namely, this result asserts the following:

> "Let (X, Σ, Q) be a probability space and let (p, q) be Hölder conjugates with $1 \leq p \leq \infty$. Let $C \subset L^p(\Sigma)$ be a $\sigma(L^p(\Sigma), L^q(\Sigma))$-closed convex cone containing $-L^p_+(\Sigma)$ with $C \cap L^p_+(\Sigma) = \{0\}$. Then there exists $y \in L^q_{++}(\Sigma)$ with $E_Q[y] = 1$ such that $E_Q[x] \leq 0$ for all $x \in C$."

Set $L^p_{\mathcal{F},+}(\mathcal{E}) := L^p_{\mathcal{F}}(\mathcal{E}) \cap L^0_+(\mathcal{E})$ and $L^p_{\mathcal{F},++}(\mathcal{E}) := L^p_{\mathcal{F}}(\mathcal{E}) \cap L^0_{++}(\mathcal{E})$. Say that a subset S of an L^0-module E is an L^0-*cone* if $\eta S \subset S$ for all $\eta \in L^0_+$. Then, the transfer principle yields the following module analogue of the Kreps-Yan theorem:

Theorem 4.1.1. *Let (p, q) be Hölder conjugates with $1 \leq p \leq \infty$. Let $S \subset L^p_{\mathcal{F}}(\mathcal{E})$ be a stable $\sigma_{\mathfrak{s}}(L^p_{\mathcal{F}}(\mathcal{E}), L^q_{\mathcal{F}}(\mathcal{E}))$-closed $L^0(\mathcal{F})$-convex $L^0(\mathcal{F})$-cone containing $-L^p_{\mathcal{F},+}(\mathcal{E})$ such that*

$$S \cap L^p_{\mathcal{F},+}(\mathcal{E}) = \{0\}.$$

Then there exists $y \in L^q_{\mathcal{F},++}(\mathcal{E})$ with $E_P[y|\mathcal{F}] = 1$ such that $E_P[x|\mathcal{F}] \leq 0$ for all $x \in S$.

Remark 4.1.2. *In the case $p = 1$, the result above can be also proven by conventional means relying on the hyperplane separation theorem [36, Theorem 2.8] and by carefully adapting to conditional expectations the proof to the Kreps-Yan theorem provided in [70, Lemma 3].*

We will use later Theorem 4.1.1 to derive a new proof for the so-called Dalang-Morton-Willinger theorem, which is the version of the First Fundamental Asset Pricing Theorem for finite markets in finite discrete time.

One more time, this is just an example of application of our transfer principle, and the number of them can be easily increased. Moreover, we will show in the next section that this amounts to a Boolean-valued transfer principle from duality theory of conventional risk measures to duality theory of conditional risk measures.

4.2 A Boolean-valued transfer principle for conditional risk measures

In what follows we will establish a transfer principle from duality theory of conventional risk measures to the duality theory of conditional risk measures.

Let us start by recalling the basic elements of duality theory of risk measures:

Suppose that our probability space $(\Omega, \mathcal{E}, P)$ models the market information at some time horizon $T > 0$. The final payoff of each financial position is going to be modelled by a subspace $\mathcal{X}$ of $L^1(\mathcal{E})$ with $\mathbb{R} \subset L^1(\mathcal{E})$ which is solid, that is, $y \in \mathcal{X}$ and $|x| \leq |y|$ imply that $x \in \mathcal{X}$. Solid subspaces are also called *order ideals*.

The risk arisen from any final payoff $x \in \mathcal{X}$ is measured by a convex function $\rho : \mathcal{X} \to \mathbb{R}$ satisfying for all $x, y \in \mathcal{X}$:

1. *Monotonicity*: i.e. $x \leq y$ implies $\rho(y) \leq \rho(x)$;
2. *Cash-invariance*: i.e. $\rho(x + r) = \rho(x) - r$ for all $r \in \mathbb{R}$.

A convex function $\rho : \mathcal{X} \to \mathbb{R}$ satisfying 1 and 2 above is called a *convex risk measure*.

A convex risk measure $\rho : \mathcal{X} \to \mathbb{R}$ is said to have

- the *Fatou property* if

$$\lim_n x_n = x \text{ a.s., } y \in \mathcal{X},\ |x_n| \leq y \text{ for all } n \in \mathbb{N} \text{ implies } \liminf_n \rho(x_n) \geq \rho(x);$$

- the *Lebesgue property* if

$$\lim_n x_n = x \text{ a.s., } y \in \mathcal{X},\ |x_n| \leq y \text{ for all } n \in \mathbb{N} \text{ implies } \liminf_n \rho(x_n) = \rho(x).$$

The Köthe dual space of $\mathcal{X}$ is defined by

$$\mathcal{X}^x := \{y \in L^0(\mathcal{E}) \colon xy \in L^1(\mathcal{E}) \text{ for all } x \in \mathcal{X}\}.$$

Then $\mathcal{X}^x$ is also a solid subspace of $L^1(\mathcal{E})$ with $\mathbb{R} \subset \mathcal{X}^x$.

In addition, we can consider the dual pair $\langle \mathcal{X}, \mathcal{X}^x \rangle$ associated to the bilinear form $(x, y) \mapsto E_P[xy]$ and the weak topologies $\sigma(\mathcal{X}, \mathcal{X}^x)$ and $\sigma(\mathcal{X}^x, \mathcal{X})$.

The conjugate of a convex risk measure ρ is defined to be

$$\rho^*(y) := \sup\{E_P[xy] - \rho(x) \colon y \in \mathcal{X}^x\}.$$

Also, we define

$$\mathcal{X}_0^x := \{y \in \mathcal{X}^x \colon E_P[y] = -1,\ y \leq 0\}.$$

Notice, that due to the Radon-Nikodym theorem, for any $y \in \mathcal{X}_0^x$, one has that $y \in \mathcal{X}_0^x$ if and only if there exists a probability measure Q which is absolutely continuous w.r.t. P such that $-y = \frac{dQ}{dP}$.

In the following, we will recall the elements of the duality theory of conditional risk measures.

Now, we suppose that $\mathcal{F}$ is a sub-σ-algebra of $\mathcal{E}$ which models the available market information at some future date $0 < t < T$. The model space of final payoffs at time T is going to be modelled by a stable solid L^0-submodule $\mathscr{X}$ of $L^1_{\mathcal{F}}(\mathcal{E})$ with $L^0(\mathcal{F}) \subset \mathscr{X}$. Then a *conditional risk measure* is an $L^0(\mathcal{F})$-convex function $\rho : \mathscr{X} \to L^0(\mathcal{F})$ satisfying:

1. *Monotonicity*: i.e. if $x \leq y$ in $\mathscr{X}$, then $\rho(x) \geq \rho(y)$;
2. $L^0(\mathcal{F})$-*cash invariance*: i.e. $\rho(x+\eta) = \rho(x) - \eta$ whenever $\eta \in L^0(\mathcal{F})$ and $x \in \mathscr{X}$;

A conditional risk measure $\rho : \mathscr{X} \to L^0(\mathcal{F})$ is said to have:

- The *Fatou property* if

$$\lim_n x_n = x \text{ a.s.}, \; y \in \mathscr{X}, \; |x_n| \leq y \text{ for all } n \in \mathbb{N} \text{ implies } \liminf_n \rho(x_n) \geq \rho(x);$$

- The *Lebesgue property* if for any sequence $(x_n) \subset \mathscr{X}$

$$\lim_n x_n = x \text{ a.s.}, \; y \in \mathscr{X}, \; |x_n| \leq y \text{ for all } n \in \mathbb{N} \text{ implies } \lim_n \rho(x_n) = \rho(x) \text{ a.s.}.$$

We define the *Köthe dual* $L^0(\mathcal{F})$*-module of* $\mathscr{X}$ to be

$$\mathscr{X}^x := \left\{ y \in L^0(\mathcal{E}) \colon xy \in L^1_{\mathcal{F}}(\mathcal{E}) \text{ for all } x \in \mathscr{X} \right\}.$$

It is simple to verify that $\mathscr{X}^x$ enjoys the same properties as $\mathscr{X}$. Namely, $\mathscr{X}^x$ is a stable solid $L^0(\mathcal{F})$-submodule of $L^1_{\mathcal{F}}(\mathcal{E})$ with $L^0(\mathcal{F}) \subset \mathscr{X}^x$.

We can consider the dual system of $L^0(\mathcal{F})$-modules $\langle \mathscr{X}, \mathscr{X}^x \rangle$ associated to the the bi-L^0-module homomorhism $(x,y) \mapsto E_P[xy|\mathcal{F}]$ and the corresponding stably weak topologies $\sigma_{\mathfrak{s}}(\mathscr{X}, \mathscr{X}^x)$ and $\sigma_{\mathfrak{s}}(\mathscr{X}^x, \mathscr{X})$.

The conjugate of a conditional risk measure is given by

$$\rho^*(y) := \sup\{E_P[xy|\mathcal{F}] - \rho(x) \colon y \in \mathscr{X}^x\}.$$

In addition, we define

$$\mathscr{X}^x_0 := \{y \in \mathscr{X}^x \colon E_P[y|\mathcal{F}] = -1, \; y \leq 0\}.$$

Notice, that due to the Radon-Nikodym theorem, for any $y \in \mathscr{X}^x$, one has that $y \in \mathscr{X}^x_0$ if and only if there exists a probability measure Q which is absolutely continuous w.r.t. P with $Q|_{\mathcal{F}} = P|_{\mathcal{F}}$ and such that $-y = \frac{dQ}{dP}$.

Next, we will study the ascent of the different elements of duality theory of conditional risk measures and how they are interpreted within the Boolean-valued model:

By construction, we can find a name $(\Omega, \Sigma, \mathbb{P})_{\mathcal{A}}$ for a probability space such that

$$[\![L^1_{\mathcal{F}}(\mathcal{E})\!\uparrow = L^1(\Sigma)]\!] = I.$$

Let $\rho : \mathscr{X} \to L^0(\mathcal{F})$ be a conditional risk measure:

- Since $\mathscr{X}$ is stable, we can consider $\mathscr{X}\!\uparrow$. Using Theorem 2.1.4, it can be verify that $\mathscr{X}\!\uparrow$ is a name for a solid vector subspace with $[\![\mathbb{R} \subset \mathscr{X}\!\uparrow \subset L^1(\Sigma)]\!] = I$.
- $\mathscr{X}^x$ is also stable and we can thus consider $\mathscr{X}^x\!\uparrow$, which satisfies $[\![\mathscr{X}^x\!\uparrow = (\mathscr{X}\!\uparrow)^x]\!] = I$.

- Since ρ is $L^0(\mathcal{F})$-convex, we can consider a name $\rho{\uparrow}$ for a function from $\mathscr{X}{\uparrow}$ to $\mathbb{R}_{\mathcal{A}}$. Then $\rho{\uparrow}$ is a name for a convex risk measure (see Proposition 2.3.8).
- Due to Proposition 4.1.7, ρ has the Fatou property (resp. the Lebesgue property) if and only if $[\![\rho{\uparrow}$ has the Fatou property$]\!] = I$ (resp. $[\![\rho{\uparrow}$ has the Lebesgue property$]\!] = I$).
- ρ is $\sigma_{\mathfrak{s}}(\mathscr{X}, \mathscr{X}^x)$-l.s.c. if and only if $[\![\rho{\uparrow}$ is $\sigma(\mathscr{X}{\uparrow}, \mathscr{X}^x{\uparrow})$-l.s.c.$]\!] = I$ (see 1 of Proposition 2.3.11).
- ρ is $\sigma_{\mathfrak{s}}(\mathscr{X}, \mathscr{X}^x)$-stably inf-compact if and only if $[\![\rho{\uparrow}$ is $\sigma(\mathscr{X}{\uparrow}, \mathscr{X}^x{\uparrow})$-inf-compact$]\!] = I$ (see 2 of Proposition 2.3.11).
- ρ^* has the local property, we can thus consider $\rho^*{\uparrow}$, which satisfies $[\![\rho^*{\uparrow} = (\rho{\uparrow})^*]\!] = I$ (see Proposition 2.3.18).
- ρ^* is $\sigma_{\mathfrak{s}}(\mathscr{X}^x, \mathscr{X})$-l.s.c. if and only if $[\![\rho^*{\uparrow}$ is $\sigma(\mathscr{X}^x{\uparrow}, \mathscr{X}{\uparrow})$-l.s.c.$]\!] = I$.
- ρ^* is $\sigma_{\mathfrak{s}}(\mathscr{X}^x, \mathscr{X})$-stably inf-compact if and only if $[\![\rho^*{\uparrow}$ is $\sigma(\mathscr{X}^x{\uparrow}, \mathscr{X}{\uparrow})$-inf-compact$]\!] = I$.
- $\mathscr{X}^x_0$ is stable. Then one can consider $\mathscr{X}^x_0{\uparrow}$, which satisfies $[\![\mathscr{X}^x_0{\uparrow} = (\mathscr{X}{\uparrow})^x_0]\!] = I$.
- Due to Proposition 2.3.10, for any $x \in \mathscr{X}$, we have the following:

$$\rho(x) = \sup\{E_P[xy|\mathcal{F}] - \rho^*(y) \colon y \in \mathscr{X}^x_0\},$$

if and only if

$$[\![\rho{\uparrow}(x^\bullet) = \sup\{E_{\mathbb{P}}[x^\bullet y] - \rho^*{\uparrow}(y) \colon y \in \mathscr{X}^x_0{\uparrow}\}]\!] = I.$$

Comments 4.2.1. *The notion of convex risk measure was independently introduced by Föllmer and Schied [39] and Fritelli and Gianin [45] as an extension of the notion of* coherent risk measure *introduced by Artzner et al. [5].*

Risk measures in a conditional setting were first time studied independently by Defensen and Scandolo [31] and by Bion-Nadal [16]. They consider functions $\rho : L^p(\mathcal{E}) \to L^p(\mathcal{F})$.

Conditional risk measure defined on L^p type modules were first time introduced by Filipovic, Kupper and Vogelpoth [37].

4.3 A general theorem of robust representation of conditional risk measures

In this section, as an instance of application of the transfer principle established in the previous section, we obtain a general robust representation result for conditional risk measures.

Suppose that $\rho : \mathcal{X} \to \mathbb{R}$ is a convex risk measure. Due to the classical Fenchel-Moreau duality theorem applied to the pairing $\langle \mathcal{X}, \mathcal{X}^x \rangle$ one has that if ρ is $\sigma(\mathcal{X}, \mathcal{X}^x)$-l.s.c., then

$$\rho(x) = \sup\{E_P[xy] - \rho^*(y) \colon y \in \mathcal{X}^x\}. \tag{4.7}$$

It is simple to verify that the converse assertion is also true.

Further, one has that $\mathrm{dom}(\rho^*) = \mathcal{X}^x_0$. Indeed, take any $y \in \mathcal{X}^x$ and fix $n \in \mathbb{N}$. Then $\rho^*(y) \geq E_P[ny^+] - \rho(n1_{\{y \geq 0\}}) \geq nE_P[y^+] - \rho(0)$. Since $n \in \mathbb{N}$ is arbitrary, $\rho^*(y) < \infty$ only if $y^+ = 0$. We also have $\rho^*(y) \geq E_P[ny] - \rho(n) \geq n(E_P[ny] + 1) - \rho(0)$. Being $n \in \mathbb{N}$ arbitrary, we conclude the $\rho^*(y) < \infty$ only if $E_P[y] = -1$.

Then the following statement holds:

"A convex risk measure $\rho : \mathcal{X} \to \mathbb{R}$ is $\sigma(\mathcal{X}, \mathcal{X}^x)$-l.s.c. if and only if

$$\rho(x) = \sup\{E_P[xy] - \rho^*(y) \colon y \in \mathcal{X}_0^x\} \quad \text{for all } x \in \mathcal{X}."$$

The following statement is proven in [89, Theorem 1.1]):

"Let $\rho : \mathcal{X} \to \mathbb{R}$ be a convex risk measure which admits a representation

$$\rho(x) = \sup\{E_P[xy] - \rho^* \colon y \in \mathcal{X}_0^x\} \quad \forall x \in \mathcal{X}.$$

The following are equivalent:

1. For each $x \in \mathcal{X}$ there exists $y \in \mathcal{X}_0^x$ such that
$$\rho(x) = E_P[xy] - \rho^*(y);$$
2. ρ has the Lebesgue property;
3. ρ^* is $\sigma(\mathcal{X}^x, \mathcal{X})$-inf-compact."

Thus, the connections provided in Section 4.2 prove the following statements:

Theorem 4.3.1. *Let $(\Omega, \mathcal{E}, P)$ be a probability space, $\mathcal{F} \subset \mathcal{E}$ a sub-σ-algebra, and $\mathscr{X}$ a stable solid $L^0(\mathcal{F})$-submodule of $L^1_{\mathcal{F}}(\mathcal{E})$ with $L^0(\mathcal{F}) \subset \mathscr{X}$. Suppose that $\rho : \mathscr{X} \to L^0(\mathcal{F})$ is a conditional risk measure. Then, the following conditions are equivalent:*

1. *ρ admits a representation*
$$\rho(x) = \sup\{E_P[xy|\mathcal{F}] - \rho^*(y) \colon y \in \mathscr{X}^x,\ y \leq 0,\ E_P[y|\mathcal{F}] = -1\} \quad \forall x \in \mathscr{X}; \tag{4.8}$$
2. *ρ is $\sigma_{\mathfrak{s}}(\mathscr{X}, \mathscr{X}^x)$-lower semi-continuous.*

Theorem 4.3.2. *Let $(\Omega, \mathcal{E}, P)$ be a probability space, $\mathcal{F} \subset \mathcal{E}$ a sub-σ-algebra, and $\mathscr{X}$ a stable solid $L^0(\mathcal{F})$-submodule of $L^1_{\mathcal{F}}(\mathcal{E})$ with $L^0(\mathcal{F}) \subset \mathscr{X}$. Suppose that $\rho : \mathscr{X} \to L^0(\mathcal{F})$ is a conditional risk measure which admits a representation*

$$\rho(x) = \sup\{E_P[xy|\mathcal{F}] - \rho^*(y) \colon y \in \mathscr{X}^x,\ y \leq 0,\ E_P[y|\mathcal{F}] = -1\}, \tag{4.9}$$

for any $x \in \mathscr{X}$. Then, the following conditions are equivalent:

1. *The representation (4.9) is attained for each $x \in \mathscr{X}$, i.e. for every $x \in \mathscr{X}$ there exists $y \in \mathscr{X}^x$ with $y \leq 0$ and $E_P[y|\mathcal{F}] = -1$ such that*
$$\rho(x) = E_P[xy|\mathcal{F}] - \rho^*(y);$$
2. *ρ has the Lebesgue property;*
3. *ρ^* is $\sigma_{\mathfrak{s}}(\mathscr{X}^x, \mathscr{X})$-stably inf-compact.*

Remark 4.3.1. *Notice that, unlikely the one-period case, the statement above does not involve classical compactness and instead it is provided a condition based on stable compactness. Recall that in Chapter 1 we proved that any Hausdorff locally stable L^0-compact topology is anti-compact when the underlying probability space is atomless (see Theorem 1.4.2). This shows that classical compactness does not make possible a robust representation theorem with full generality and stable compactness appears to be a suitable substitute for compactness is this setting.*

Notice that Theorem 4.3.1 can be proven by using the module analogue of the Fenchel-Moreau theorem provided in [36] (see Theorem 2.4.3).

Next, we will see the form of Theorems 4.3.1 and 4.3.2 in some particular cases.

4.3.1 A Jouini-Schachermayer-Touzi theorem for conditional risk measures

The Köthe dual space of $L^\infty(\mathcal{E})$ is $L^1(\mathcal{E})$. If $\mathscr{X} = L^\infty_{\mathcal{F}}(\mathcal{E})$, then $\mathscr{X}^x = L^1_{\mathcal{F}}(\mathcal{E})$ due to the transfer principle. Delbaen [28] proved the following:[1]

> "A convex risk measure $\rho : L^\infty(\mathcal{E}) \to \mathbb{R}$ has the Fatou property if and only if it admits a representation
>
> $$\rho(x) = \sup\{E_P[xy] - \rho^*(y) \colon y \in L^1(\mathcal{E}),\ y \leq 0,\ E_P[y] = -1\}$$
>
> for all $x \in L^\infty(\mathcal{E})$."

On the other hand, the so-called Jouini-Schachermayer-Touzi theorem (see [29, Theorem 2] and for its original form see [68]) asserts the following:

> "Let $\rho : L^\infty(\mathcal{E}) \to \mathbb{R}$ be a convex risk measure with the Fatou property. The following are equivalent:
>
> 1. For each $x \in L^\infty(\mathcal{E})$ there exists $y \in L^1(\mathcal{E})$ with $y \leq 0$ and $E_P[y] = -1$ such that
> $$\rho(x) = E_P[xy] - \rho^*(y);$$
> 2. ρ has the Lebesgue property;
> 3. ρ^* is $\sigma(L^1(\mathcal{E}), L^\infty(\mathcal{E}))$-inf-compact."

Then, the transfer principle yields the following:

Theorem 4.3.3. *Let $\rho : L^\infty_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ be a conditional risk measure. The following are equivalent:*

1. *ρ admits a representation*
$$\rho(x) = \sup\{E_P[xy|\mathcal{F}] - \rho^*(y) \colon y \in L^1_{\mathcal{F}}(\mathcal{E}),\ y \leq 0,\ E_P[y|\mathcal{F}] = -1\} \quad \forall x \in L^\infty_{\mathcal{F}}(\mathcal{E});$$
2. *ρ has the Fatou property;*
3. *ρ is $\sigma_{\mathfrak{s}}(L^\infty_{\mathcal{F}}(\mathcal{E}), L^1_{\mathcal{F}}(\mathcal{E}))$-lower semi-continuous.*

Theorem 4.3.4. *Let $\rho : L^\infty_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ be a conditional risk measure with the Fatou property. Then, the following are equivalent:*

1. *For every $x \in L^\infty_{\mathcal{F}}(\mathcal{E})$ there exists $y \in L^1_{\mathcal{F}}(\mathcal{E})$ with $y \leq 0$ and $E[y|\mathcal{F}] = -1$ such that*
$$\rho(x) = E_P[xy|\mathcal{F}] - \rho^*(y);$$
2. *ρ has the Lebesgue property;*
3. *ρ^* is $\sigma_{\mathfrak{s}}(L^1_{\mathcal{F}}(\mathcal{E}), L^\infty_{\mathcal{F}}(\mathcal{E}))$-stably inf-compact.*

[1] This was originally proven for the particular case when ρ is a coherent risk measures in [28]. Later Föllmer et al. [40] proved the general case.

We claim that, in both theorems above, the $L^0(\mathcal{F})$-convexity of ρ can be relaxed assuming instead that ρ is convex (i.e. considering scalar in the real numbers). In fact, we have the following:

Proposition 4.3.1. *Let $\rho : L^\infty_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ be convex and satisfying monotonicity and $L^0(\mathcal{F})$-cash-invariance. Then ρ is $L^0(\mathcal{F})$-norm Lipschitz continuous: i.e.*

$$|\rho(x) - \rho(y)| \leq \|x - y|\mathcal{F}\|_\infty \quad \textit{for all } x, y \in L^\infty_{\mathcal{F}}(\mathcal{E}). \tag{4.10}$$

In addition, ρ has the local property and is $L^0(\mathcal{F})$-convex.

Proof. For given $x, y \in L^\infty_{\mathcal{F}}(\mathcal{E})$, one has that $x \leq y + \|x - y|\mathcal{F}\|_\infty$. My monotonicity and $L^0(\mathcal{F})$-cash-invariance it follows that $\rho(y) - \rho(x) \leq \|x - y|\mathcal{F}\|_\infty$. By reversing the roles of x and y, we obtain (4.10).

To show the local property, fix $a \in \bar{\mathcal{F}}$ and $x \in L^\infty_{\mathcal{F}}(\mathcal{E})$. Then due to (4.10)

$$|1_a\rho(x) - 1_a\rho(1_a x)| = 1_a|\rho(x) - \rho(1_a x)| \leq 1_a\|x - 1_a x|\mathcal{F}\|_\infty = \|1_a x - 1_a x|\mathcal{F}\|_\infty = 0.$$

To prove the $L^0(\mathcal{F})$-convexity, fix $\eta \in L^0$ with $0 \leq \eta \leq 1$ and $x, y \in L^\infty_{\mathcal{F}}(\mathcal{E})$. One has that $\eta = \sup\{\mathfrak{q} \in L^0(\mathbb{Q})\colon 0 \leq \mathfrak{q} \leq \eta\}$. For each $n \in \mathbb{N}$, we can find $\mathfrak{q}_n \in L^0(\mathbb{Q})$ with $0 \vee (\eta - 1/n) \leq \mathfrak{q}_n \leq \eta$. For $\mathfrak{n} \in L^0(\mathbb{N})$, define $\mathfrak{q}_\mathfrak{n} := \sum_k 1_{\{\mathfrak{n}=k\}}\mathfrak{q}_k$. Then $(\mathfrak{q}_\mathfrak{n})_{\mathfrak{n} \in L^0(\mathbb{N})}$ is a stable sequence which $|\cdot|$-converges to η. Since $(\mathfrak{q}_\mathfrak{n}) \subset \mathfrak{s}(\mathbb{Q})$ and ρ has the local property, one has

$$\rho(\mathfrak{q}_\mathfrak{n} x + (1 - \mathfrak{q}_\mathfrak{n})y) \leq \mathfrak{q}_\mathfrak{n}\rho(x) + (1 - \mathfrak{q}_\mathfrak{n})\rho(y) \quad \text{for all } \mathfrak{n} \in L^0(\mathbb{N}).$$

Due to (4.10), ρ is continuous, taking limits above, we obtain

$$\rho(\eta x + (1 - \eta)y) \leq \eta\rho(x) + (1 - \eta)\rho(y).$$

$\square$

Comments 4.3.1. *Theorem 4.3.3 is already known in literature; as shown in [55] it can be easily obtained from the earlier versions [31, 16].*

A proof of Theorem 4.3.4 by conventional means can be found in Section B.6 in the Appendix, which is based on the conditional perturbed James' compactness theorem B.5.1 which is also proven by conventional means in Section B.5 in the Appendix.

4.3.2 Conditional risk measures on L^p type modules for $1 \leq p < \infty$

Let (p, q) be Hölder conjugates with $1 \leq p < \infty$. It is known that the Köthe dual space of L^p is precisely L^q (see eg [114, Example 29.4]). Suppose that $\mathscr{X} := L^p_{\mathcal{F}}(\mathcal{E})$. Due to the transfer principle, we have $\mathscr{X}^x = L^q_{\mathcal{F}}(\mathcal{E})$.

It is known that any convex risk measure $\rho : L^p \to \mathbb{R}$ has the Lebesgue property and admits a dual representation which attains (see eg [71, Theorem 2.11]). Thus, by the transfer principle we have the following:

Theorem 4.3.5. *Let $\rho : L^p_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ be a conditional risk measure. Then, ρ has the Lebesgue property, ρ^* is $\sigma_\mathfrak{s}(L^q_{\mathcal{F}}(\mathcal{E}), L^p_{\mathcal{F}}(\mathcal{E}))$-stably inf-compact, and for every $x \in L^p_{\mathcal{F}}(\mathcal{E})$ it holds:*

$$\rho(x) = \max\{E_P[xy|\mathcal{F}] - \rho^*(y)\colon y \in L^q_{\mathcal{F}}(\mathcal{E}),\ y \leq 0,\ E_P[y|\mathcal{F}] = -1\}.$$

Comments 4.3.2. *Theorem 4.3.1 in the case of conditional risk measures on L^p type modules $(1 \leq p < \infty)$ is already known as it was obtained in [37]. It was proven by means of the module analogue of the Fenchel-Moreau theorem provided in [36] (see Theorem 2.4.3).*

4.3.3 Conditional risk measures on Orlicz type modules

In the recent literature we can found many works that deal with risk measures defined on Orlicz and Orlicz heart spaces, cf. [3, 24, 87].

Next, we will see the form of Theorems 4.3.2 and 4.3.2 in the case of modules of the type Orlicz. We will see the Orlicz-heart case in the next subsection.

Let (ϕ,ψ) be Young conjugate functions, that is, $\psi(r)=\sup\{rs-\phi(s)\colon \phi(s)\}$. It is well-known that the Köthe dual space of L^ϕ is precisely L^ψ (see eg [83, 113]). Suppose that $\mathscr{X}:=L^\phi_\mathcal{F}(\mathcal{E})$. In this case we have $\mathscr{X}^x=L^\psi_\mathcal{F}(\mathcal{E})$, due to the transfer principle.

In this case, Theorems 4.3.1 and 4.3.1 apply directly and yield the following:

Theorem 4.3.6. *Let $\rho : L^\phi_\mathcal{F}(\mathcal{E})\to L^0(\mathcal{F})$ a conditional risk measure. The following are equivalent:*

1. *ρ admits a representation*

$$\rho(x)=\sup\{E_P[xy|\mathcal{F}]-\rho^*(y)\colon y\in L^\psi_\mathcal{F}(\mathcal{E}),\ y\leq 0,\ E_P[y|\mathcal{F}]=-1\}\quad \forall x\in L^\phi_\mathcal{F}(\mathcal{E}); \tag{4.11}$$

2. *ρ is $\sigma_\mathfrak{s}(L^\phi_\mathcal{F}(\mathcal{E}),L^\psi_\mathcal{F}(\mathcal{E}))$-lower semi-continuous.*

Theorem 4.3.7. *Let $\rho : L^\phi_\mathcal{F}(\mathcal{E})\to L^0(\mathcal{F})$ a conditional risk measure and suppose that ρ admits a representation*

$$\rho(x)=\sup\{E_P[xy|\mathcal{F}]-\rho^*(y)\colon y\in L^\psi_\mathcal{F}(\mathcal{E}),\ y\leq 0,\ E_P[y|\mathcal{F}]=-1\}\quad \forall x\in L^\phi_\mathcal{F}(\mathcal{E}). \tag{4.12}$$

Then, the following are equivalent:

1. *The representation (4.12) is attained for each $x\in L^\phi_\mathcal{F}(\mathcal{E})$, i.e. for every $x\in L^\phi_\mathcal{F}(\mathcal{E})$ there exists $y\in L^\psi_\mathcal{F}(\mathcal{E})$ with $y\leq 0$ and $E[y|\mathcal{F}]=-1$ such that*

$$\rho(x)=E_P[xy|\mathcal{F}]-\rho^*(y);$$

2. *ρ has the Lebesgue property;*
3. *ρ^* is $\sigma_\mathfrak{s}(L^\psi_\mathcal{F}(\mathcal{E}),L^\phi_\mathcal{F}(\mathcal{E}))$-stably inf-compact.*

4.3.4 Conditional risk measures on Orlicz-heart type modules

Let (ϕ,ψ) be Young conjugate functions with ϕ finite-valued. It is known that the Köthe dual space of H^ϕ is $L^\psi_\mathcal{F}$ (see eg [89]). Suppose that $\mathscr{X}:=H^\phi_\mathcal{F}(\mathcal{E})$. In this case we have $\mathscr{X}^x=L^\psi_\mathcal{F}(\mathcal{E})$.

It is known that any convex risk measure $\rho : H^\phi\to\mathbb{R}$ has the Lebesgue property (see eg [24, Theorem 4.4]). Thus, due to the transfer principle we have the following:

Theorem 4.3.8. *Let $\rho : H^\phi_\mathcal{F}(\mathcal{E})\to L^0(\mathcal{F})$ be a conditional risk measure. Then, ρ has the Lebesgue property, ρ^* is $\sigma_\mathfrak{s}(L^\psi_\mathcal{F}(\mathcal{E}),H^\phi_\mathcal{F}(\mathcal{E}))$-stably inf-compact, and for every $x\in H^\phi_\mathcal{F}(\mathcal{E})$ it holds:*

$$\rho(x)=\max\{E_P[xy|\mathcal{F}]-\rho^*(y)\colon y\in L^\psi_\mathcal{F}(\mathcal{E}),\ y\leq 0,\ E_P[y|\mathcal{F}]=-1\}.$$

Chapter 5

Optimal stochastic control in discrete time

In this Chapter, we investigate new applications of L^0-convex analysis to stochastic control and mathematical finance in finite discrete time.

Specifically, we consider as a model space the L^0-module $L^0(\mathbb{R}^d)$ endowed with the module Euclidean L^0-norm. L^0-convex analysis on $L^0(\mathbb{R}^d)$ is studied in [23]. In Section 5.1, we introduce some nomenclature and recall some results of [23] that we will use in this chapter.

In Section 5.2, we prove the existence of optimal solutions for two types of parametrized stochastic control problems. By means of a suitable Bellman principle together with a recursive scheme, we reduce a multi-period problem to a one-period problem, which is solved by the module analogue of the well-known fact that every real-valued upper semi-continuous function attains its maxima on a compact set.

Section 5.3 is devoted to the study of some related mathematical finance problems in finite discrete time. Namely, we find optimal solutions for wealth-dependent utility maximization with self-financing strategies and short-selling constraints, and for wealth-dependent maximization of generalized entropic utility functions. We also find an explicit solution for a wealth-dependent dynamic utility sharing problem with generalized entropic utility functions.

A tool classically used to study stochastic control are normal integrands. In Section 5.4, we study the relations between this classical approach and the L^0-convex approach. Namely, we find that normal integrands (almost surely identified) are in one-to-one relation with lower semi-continuous functions from $L^0(\mathbb{R}^d)$ to $\bar{L}^0$ with the local property.

Finally, in Section 5.5, by using L^0-convex analysis, we provide a new proof for the Dalang-Morton-Willinger theorem, which is the finite-time version of the celebrated First Fundamental Theorem of Asset Pricing.

Just comment that this Chapter is mostly self-contained and does not need Boolean-valued analysis, as we use module analogues of real analysis results, which are proven in [23] by conventional means. Thus, the reader without knowledge on Boolean-valued models can follow easily this chapter.

5.1 Preliminaries

In this chapter, we will work with the L^0-module $L^0(\mathbb{R}^d)$ where d is a natural number. L^0-convex analysis in $L^0(\mathbb{R}^d)$ has been studied in [23]. In what follows, we will introduce some nomenclature and preliminary results that we will use throughout this chapter.

For any $\eta, \xi \in L^0(\mathbb{R}^d)$, $\eta = (\eta_1, \ldots, \eta_d)$ and $\xi = (\xi_1, \ldots, \xi_d)$, define $\langle \eta, \xi \rangle := \sum_{i=1}^d \eta_i \xi_i$, which is the module analogue of the Euclidean scalar product. In particular, if $\eta \in L^0(\mathbb{R}^d)$, then $|\eta| := \sqrt{\langle \eta, \eta \rangle}$ defines an L^0-norm, which is called the *Euclidean L^0-norm* and induces a locally stable L^0-convex topology on $L^0(\mathbb{R}^d)$.

Remark 5.1.1. *Let us see a suitable Boolean-valued interpretation of $L^0(\mathbb{R}^d)$. Suppose that $\mathcal{A} := \bar{\mathcal{F}}$ is the probability algebra associated to $\mathcal{F}$ and consider the model $V^{(\mathcal{A})}$. We can take the names $\check{d}$ and $\mathbb{R}_{\mathcal{A}}^{\check{d}}$. Then, in view of Proposition 2.1.3, we have that*

$$L^0(\mathbb{R}) \longrightarrow \mathbb{R}_{\mathcal{A}}^{\check{d}}\!\downarrow \quad \eta = (\eta_1, \ldots, \eta_d) \mapsto \eta^\bullet := (\eta_1^\bullet, \ldots, \eta_d^\bullet)_{\mathcal{A}},$$

is a stable bijection. This justifies the definition of the ascent $L^0(\mathbb{R}^d)\!\uparrow := \mathbb{R}_{\mathcal{A}}^{\check{d}}$. Therefore, due to the transfer principle, we can apply module analogues of known results on $\mathbb{R}^d$.

However, in the forthcoming sections, we will not deal with Boolean-valued interpretations because the module analogues that we will use throughout are known from [23], which are proven by conventional means.

Given a sequence (η_n) in $L^0(\mathbb{R}^d)$, as we have done many times, one can define a stable sequence $(\eta_\mathfrak{n})_{\mathfrak{n} \in L^0(\mathbb{N})}$ where $\eta_\mathfrak{n} = \sum_k 1_{\{\mathfrak{n}=k\}} \eta_k$. Conversely, if $(\eta_\mathfrak{n})_{\mathfrak{n} \in L^0(\mathbb{N})}$ is a stable sequence, we can consider the sequence $(\eta_n)_{n \in \mathbb{N}}$. This defines a bijection between sequences and stable sequences in $L^0(\mathbb{R}^d)$.

The following result can be derived from Proposition 2.1.5. However, we give a proof:

Proposition 5.1.1. *Suppose that (η_n) is a sequence in $L^0(\mathbb{R}^d)$, then $\lim_n \eta_n = \eta$ a.s. if and only if $\lim_\mathfrak{n} \eta_\mathfrak{n} = \eta$ w.r.t. $|\cdot|$.*

Proof. Suppose that $\lim_n \eta_n = \eta$ a.s.. Then the stable sequence $(\eta_\mathfrak{n})$ where $\eta_\mathfrak{n} = \sum_{k \in \mathbb{N}} 1_{\{\mathfrak{n}=k\}} \eta_k$ for any $\mathfrak{n} \in L^0(\mathbb{N})$ $|\cdot|$-converges to η. Indeed, for each $\mathfrak{n} \in L^0(\mathbb{N})$ define $\xi_\mathfrak{n} := \sup_{\mathfrak{m} \geq \mathfrak{n}} |\eta_\mathfrak{m} - \eta|$. We have that

$$\inf_\mathfrak{n} \xi_\mathfrak{n} = \inf_\mathfrak{n} \sup_{\mathfrak{m} \geq \mathfrak{n}} |\eta_\mathfrak{m} - \eta| = \inf_n \sup_{m \geq n} |\eta_m - \eta| = \limsup_n |\eta_n - \eta| = 0.$$

Given $\varepsilon \in L^0_{++}$, due to Lemma 1.2.1, there exists $\mathfrak{n}_\varepsilon \in L^0(\mathbb{N})$ such that

$$\mathfrak{n} \geq \mathfrak{n}_\varepsilon \quad \text{implies} \quad \xi_\mathfrak{n} \leq \varepsilon.$$

This yields the conclusion.

Suppose that $\lim_\mathfrak{n} |\eta_\mathfrak{n} - \eta| = 0$. For each $k \in \mathbb{N}$, there exists $\mathfrak{n}_k \in L^0(\mathbb{N})$ such that

$$|\eta_\mathfrak{n} - \eta| \leq \frac{1}{k} \quad \text{for all } \mathfrak{n} \geq \mathfrak{n}_k.$$

Then

$$0 \leq \limsup_n |\eta_n - \eta| = \inf_n \sup_{m \geq n} |\eta_n - \eta| = \inf_\mathfrak{n} \sup_{\mathfrak{m} \geq \mathfrak{n}} |\eta_\mathfrak{n} - \eta| \leq \sup_{\mathfrak{m} \geq \mathfrak{n}_k} |\eta_\mathfrak{n} - \eta| \leq \frac{1}{k}.$$

Since $k \in \mathbb{N}$ is arbitrary, we conclude that $\limsup_n |\eta_n - \eta| = 0$.

□

Definition 5.1.1. *A subset S of $L^0(\mathbb{R}^d)$ is said to be* sequentially closed *if $\eta \in S$ whenever $(\eta_n) \subset S$ and $\lim_n \eta_n = \eta$ a.s..*

As a consequence of Proposition 5.1.1, we have the following:

Corollary 5.1.1. *A stable subset S of $L^0(\mathbb{R}^d)$ is $|\cdot|$-closed if and only if it is sequentially closed.*

The next result is known (see eg [70, Lemma 2], [41, Lemma 1.63] or [23, Theorem 3.8]). Also, it is the module analogue of the Bolzano-Weierstrass theorem.

Proposition 5.1.2. *Suppose that (η_n) is a sequence in $L^0(\mathbb{R}^d)$ such that $\sup_n |\eta_n| < \infty$, then there exist a sequence $\mathfrak{n}_1 < \mathfrak{n}_2 < \ldots$ in $L^0(\mathbb{N})$ and $\eta \in L^0(\mathbb{R}^d)$ such that $\lim_k \eta_{\mathfrak{n}_k} = \eta$ a.s.. In addition, given $\varepsilon \in L^0_{++}$ with $|\eta| > \varepsilon$ it is possible to construct $(\eta_{\mathfrak{n}_k})$ so that $|\eta_{\mathfrak{n}_k}| \geq \varepsilon$ for all $k \in \mathbb{N}$.*

The following result is related to the proposition above. It is the module analogue of the fact that every sequence of real numbers has a subsequence which converges to its upper limit. It can be proven by adapting the argument that proves the latter result.

Proposition 5.1.3. *If (η_n) is a sequence in L^0, then there exists a sequence $\mathfrak{n}_1 < \mathfrak{n}_2 < \ldots$ in $L^0(\mathbb{N})$ such that $\lim_k \eta_{\mathfrak{n}_k} = \limsup_n \eta_n$ a.s.. Moreover, for a given $\varepsilon \in L^0_{++}$ with $\varepsilon < \limsup_n \eta_n$, we can construct $(\eta_{\mathfrak{n}_k})$ so that $\eta_{\mathfrak{n}_k} > \varepsilon$ for all $k \in \mathbb{N}$.*

Definition 5.1.2. *Let S be a sequentially closed subset of $L^0(\mathbb{R}^d)$. A function $f : S \to \bar{L}^0([-\infty, \infty))$ is said to be* sequentially upper semi-continuous *if*

$$\limsup_n f(x_n) \leq f(x) \quad \text{whenever } \lim_n x_n = x \textit{ a.s. in } S.$$

Given a function $f : S \to \bar{L}^0$, with $S \subset L^0(\mathbb{R}^d)$, we define the ξ-super level set to be $V^\xi(f) := \{\eta \in L^0(\mathbb{R}^d) \colon f(x) \geq \eta\}$ for $\xi \in \bar{L}^0$.

Proposition 5.1.4. *Let S be stable and sequentially closed and $f : L^0((a,\infty)^d) \to \bar{L}^0([-\infty,\infty))$ a stable function. The following are equivalent:*

1. *$V^\xi(f)$ is $|\cdot|$-closed for all $\xi \in \bar{L}^0$;*
2. *$V^\xi(f)$ is sequentially closed for all $\xi \in \bar{L}^0$;*
3. *f is sequentially upper semi-continuous.*

If, moreover, f is increasing, then the conditions above are also equivalent to

$$\eta_n \searrow \eta \textit{ a.s. in } S \textit{ implies } f(\eta_n) \searrow f(\eta) \textit{ a.s.}.$$

Proof. $1 \Leftrightarrow 2$ is consequence of Corollary 5.1.1.

$2 \Rightarrow 3$: Let (η_n) be a sequence with $\lim_n \eta_n = \eta$ a.s. in $L^0(\mathbb{R}^d)$. Let $\xi < \limsup_n f(\eta_n)$. Due to Proposition 5.1.3, one can take a sequence $\mathfrak{n}_1 < \mathfrak{n}_2 < \ldots$ in $L^0(\mathbb{N})$ such that $\lim_k f(\eta_{\mathfrak{n}_k}) = \limsup_n f(\eta_n)$ and so that $f(\eta_{\mathfrak{n}_k}) > \xi$ for all k, and thus $(\eta_{\mathfrak{n}_k}) \subset V^\xi(f)$. Since $\lim_k \eta_{\mathfrak{n}_k} = \eta$ a.s. and $V^\xi(f)$ is sequentially closed, one has that $\eta \in V^\xi(f)$ and thus $f(\eta) \geq \xi$. Since ξ is arbitrary, necessarily $f(\eta) \geq \limsup_n f(\eta_n)$.

$3 \Rightarrow 2$: To reach a contradiction, suppose that there exists $(\eta_n) \subset V^\xi(f)$ with $\lim_n \eta_n = \eta$ a.s., $\xi \in L^0$ and $f(\eta) < \xi$ on some $a \in \mathcal{A} \setminus \{0\}$. Then, since $f(\eta_n) \geq \xi$, one has

$$f(\eta) \leq \limsup_n f(\eta_n) \leq \xi < f(\eta) \quad \text{on } a,$$

which is a contradiction.

Finally, suppose that f is increasing. If f is upper semi-continuous and $\eta_n \searrow \eta$ a.s., then, since f is s.u.s.c., we have that $f(\eta) \geq \limsup_n f(\eta_n)$; since f is increasing we have that $\liminf_n f(\eta_n) \geq f(\eta)$. We conclude that $f(\eta_n) \searrow f(\eta)$ a.s..

Conversely, suppose that $f(\eta_n) \searrow f(\eta)$ a.s. whenever $\eta_n \searrow \eta$ a.s.. If $\lim_n \eta_n = \eta$ a.s., then $\xi_n \searrow \eta$ a.s. where $\xi_n := \sup_{m \geq n} \eta < \infty$. Therefore $f(\xi_n) \searrow f(\eta)$ a.s.. Besides, since f is increasing, one has $f(\eta_n) \leq f(\xi_n)$, each n. Then

$$\limsup_n f(\eta_n) \leq \lim_n f(\xi_n) = f(\eta).$$

□

The next result is the module analogue of the elementary fact that every upper semi-continuous function attains a maxima on a compact set (see Theorem 2.4.11).

Proposition 5.1.5. *[23, Theorem 4.4] Let K be sequentially closed, stable and L^0-norm bounded and suppose that $f : K \to \bar{L}^0([-\infty, \infty))$ is sequentially upper semi-continuous, then f attains a maximum on $\xi \in K$.*

The following result is the module analogue of the fact that every subspace has a orthogonal complement in a finite-dimensional euclidean space.

Proposition 5.1.6. *[23, Corollary 2.11][52, Corollary 49] Let S be a stable L^0-submodule of $L^0(\mathbb{R}^d)$ and define*

$$S^\perp := \{\eta \in L^0(\mathbb{R}^d) \colon \langle \eta, \xi \rangle = 0, \ \forall \xi \in S\}.$$

Then $S + S^\perp = L^0(\mathbb{R}^d)$, $S \cap S^\perp = \{0\}$, and $S = S^{\perp\perp}$.

The following result is the module analogue of the fact that every directionally bounded convex subset of $\mathbb{R}^d$ is bounded.

Lemma 5.1.1. *[23, Theorem 3.13] Let S be a sequentially closed L^0-convex subset of $L^0(\mathbb{R}^d)$ containing 0. Then S is L^0-norm bounded if and only if for any $x \in S \setminus \{0\}$ there exists a $k \in \mathbb{N}$ such that $kx \notin S$.*

5.2 Parametrized stochastic control problems

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, P)$ be a filtered probability space. Let us fix some $H \subset \mathbb{R}^d$. Let

$$v_t : L^0_t(H) \times L^0_t(\mathbb{R}^n) \longrightarrow L^0_{t+1}(H), \quad t = 0, \dots, T-1$$

be a *forward generator* which is

(v1) $\mathcal{F}_t$-stable: i.e. $v_t\left(\sum_k 1_{a_k}(x_k, z_k)\right) = \sum_k 1_{a_k} v_t(x_k, z_k)$ for all $(a_k) \in p_t(I)$ and $(x_k, z_k) \subset L^0_t(H) \times L^0_t(\mathbb{R}^n)$;

(v2) sequentially continuous: i.e. $\lim_k v_t(x_k, z_k) = v_t(x, z)$ a.s. whenever $\lim_k x_k = x$ a.s. and $\lim z_k = z$ a.s. in $L^0_t(H)$ and $L^0_t(\mathbb{R}^n)$, respectively.

Let

$$u_t : L^0_t(H) \times \bar{L}^0_{t+1}([-\infty, \infty)) \times L^0_t(\mathbb{R}^n) \longrightarrow \bar{L}^0_t([-\infty, \infty)), \quad t = 0, 1, \dots, T$$

be a *backward generator*, where u_T is independent of the second and third component, which is

(u1) $\mathcal{F}_t$-stable: i.e. $u_t\left(\sum_k 1_{a_k}(x_k, y_k)\right) = \sum_k 1_{a_k} u_t(x_k, y_k)$ for all $(a_k) \in p_t(I)$ and $(x_k, y_k) \subset L^0_t(H) \times \bar{L}^0_{t+1}([-\infty, \infty))$;

(u2) increasing in the second component: i.e. if $y \leq y'$, then $u_t(x, y) \leq u_t(x, y')$ for all $x \in L^0_t(H)$ and $y, y' \in \bar{L}^0_{t+1}([-\infty, \infty))$;

(u3) sequential upper semi-continuity: i.e. $\limsup_k u_t(x_k, y_k, z_k) \leq u_t(x, y, z)$ whenever $\lim_k x_k = x$ a.s., $\lim_k y_k = y$ a.s. and $\lim_k z_k = z$ a.s. in $L^0_t(H)$ and $\bar{L}^0_{t+1}([-\infty, \infty))$, respectively.

For $t = 0, \ldots, T-1$ and $x \in L^0_t(H)$, let $\emptyset \neq \Theta_t(x) \subset L^0_t(\mathbb{R}^n)$ be a *state-dependent control set* such that

(c1) the set-valued map $x \mapsto \Theta_t(x)$ is $\mathcal{F}_t$-stable: i.e.

$$\Theta_t\left(\sum_k 1_{a_k} x_k\right) = \sum_k 1_{a_k} \Theta_t(x_k) \quad \text{for all } (a_k) \in p_t(I) \text{ and } (x_k) \subset L^0_t(H);$$

(c2) the set $\{(x, z) \in L^0_t(H) \times L^0_t(\mathbb{R}^n) \colon z \in \Theta_t(x)\}$ is sequentially closed: i.e. $z \in \Theta_t(x)$ whenever $\lim_k z_k = z$ a.s. and $\lim_k x_k = x$ a.s. with $z_k \in \Theta_t(x_k)$;

(c3) for all $x \in L^0_t(H)$ there exists $M(x) \in L^0_t$ such that

a) $|z| \leq M(x)$ for all $z \in \Theta_t(x)$ (L^0_t-norm boundedness);

b) $\liminf_k M(x_k) \in L^0_t$ whenever $\lim_k x_k = x$ a.s. in $L^0_t(H)$.

For $t = 0, \ldots, T-1$ and $x_t \in L^0_t(H)$, define recursively a *forward process* $(x_s)_{s=t}^T$ by

$$x_{s+1} := v_s(x_s, z_s), \quad s = t, \ldots, T-1$$

where $(z_s)_{s=t}^{T-1}$ is a *control process* in $\Theta_t(x_t) \times \Theta_{t+1}(x_{t+1}) \times \ldots \times \Theta_{T-1}(x_{T-1})$. For a given $x_t \in L^0_t(H)$, we consider the set

$$C_t(x_t) := \left\{((x_s)_{s=t}^T, (z_s)_{s=t}^{T-1}) \colon x_{s+1} = v_s(x_s, z_s), z_s \in \Theta_s(x_s) \text{ for } s = t, \ldots, T-1\right\}.$$

Given a forward process $(x_s)_{s=t}^T$, we define a *backward process* $(y_s(x_s))_{s=t}^T$ by

$$y_s(x_s) := \sup_{((x_k)_{k=s}^T, (z_k)_{k=s}^{T-1}) \in C_s(x_s)} u_s(x_s, \cdot, z_s) \circ \ldots \circ u_{T-1}(x_{T-1}, \cdot, z_{T-1}) \circ u_T(x_T), \quad s = t, \ldots, T-1.$$

We have the following *Bellman principle*: For fixed t and $x_t \in L^0_t(H)$, suppose that

$$\sup_{z \in \Theta_t(x_t)} u_t(x_t, y_{t+1}(v_t(x_t, z)), z)$$

is attained at $z^*_t \in \Theta_t(x_t)$. Let $x^*_{t+1} := v_t(x_t, z^*_t)$ and suppose that

$$\sup_{((x_s)_{s=t+1}^T, (z_s)_{s=t+1}^{T-1}) \in C_{t+1}(x^*_{t+1})} u_{t+1}(x_{t+1}, \cdot, z_{t+1}) \circ \ldots \circ u_{T-1}(x_{T-1}, \cdot, z_{T-1}) \circ u_T(x_T)$$

is also attained at $((x^*_s)_{s=t+1}^T, (z^*_s)_{s=t+1}^{T-1}) \in C_{t+1}(x^*_{t+1})$. Then, it follows from the monotonicity assumption (u2) that

$$\begin{aligned}
&u_t(x_t, \cdot, z^*_t) \circ u_{t+1}(x^*_{t+1}, \cdot, z^*_{t+1}) \circ \ldots \circ u_{T-1}(x^*_{T-1}, \cdot, z^*_{T-1}) \circ u_T(x^*_T) \\
&= \max_{z \in \Theta_t(x_t)} u_t(x_t, \max_{C_{t+1}(v_t(x_t, z))} u_{t+1}(x_{t+1}, \cdot, z_{t+1}) \circ \ldots \circ u_{T-1}(x_{T-1}, \cdot, z_{T-1}) \circ u_T(x_T), z) \\
&= \max_{C_t(x_t)} u_t(x_t, \cdot, z_t) \circ \ldots \circ u_{T-1}(x_{T-1}, \cdot, z_{T-1}) \circ u_T(x_T) = y_t(x_t).
\end{aligned}$$

In other words, the Bellman principle reduces a multi-period optimization problem to a one-period optimization problem. This allows to prove the following:

Theorem 5.2.1. *Suppose that (v1), (v2), (u1)–(u3) and (c1)–(c3) are fulfilled. For all $t = 0,\dots,T-1$ the following assertions are satisfied:*

(i) For all $x_t \in L^0_t(H)$ there exists $((x^*_s)^T_{s=t},(z^*_s)^{T-1}_{s=t}) \in C_t(x_t)$ such that

$$\begin{aligned} y_s(x_s) &= u_s(x_s,\cdot,z^*_s)\circ\ldots\circ u_{T-1}(x^*_{T-1},\cdot,z^*_{T-1})\circ u_T(x^*_T) \\ &= u_s(x_s,y_{s+1}(v_s(x_s,z^*_s)),z^*_s), \quad s=t,\dots,T-1. \end{aligned}$$

(ii) The function $y_t\colon L^0_t(H)\to \bar L^0_t([-\infty,\infty))$ is $\mathcal{F}_t$-stable and sequentially upper semi-continuous.

Proof. The assertions are proved by backward induction. By definition, (i)-(ii) are satisfied at time T. Suppose that they are satisfied at time $t+1$ for $t=0,\dots,T-1$. We will prove (i)-(ii) at time t.

(i) Let $x_t \in L^0_t(H)$, and denote

$$f_t(x_t,\cdot)\colon \Theta_t(x_t)\longrightarrow \bar L^0_t([-\infty,\infty)),\quad f_t(x_t,z):=u_t(x_t,y_{t+1}(x_t,z),z).$$

It can be checked that $f_t(x_t,\cdot)$ is $\mathcal{F}_t$-stable. We will prove that $f_t(x_t,\cdot)$ is sequentially upper semi-continuous. First note that $\Theta_t(x_t)$ is sequentially closed due to (c2). Suppose that $\lim_k z_k = z$ a.s. in $\Theta_t(x_t)$. By (v2) and induction hypothesis,

$$\limsup_k y_{t+1}(v_t(x_t,z_k)) \le y_{t+1}(v_t(x_t,z)) < \infty. \tag{5.1}$$

By (u2), (u3) and (5.1), one obtains

$$\begin{aligned} \limsup_k f_t(x_t,z_k) &= \limsup_k u_t(x_t,y_{t+1}(v_t(x_t,z_k)),z_k) \\ &\le \limsup_k u_t(x_t,\sup_{k'\ge k} y_{t+1}(v_t(x_t,z_{k'})),z_k) \\ &\le u_t(x_t,\lim_k \sup_{k'\ge k} y_{t+1}(v_t(x_t,z_{k'})),z) \\ &= u_t(x_t,\limsup_k y_{t+1}(v_t(x_t,z_k)),z) \\ &\le u_t(x_t,y_{t+1}(v_t(x_t,z)),z) = f_t(x_t,z). \end{aligned}$$

This proves that $f_t(x_t,\cdot)$ is sequentially upper semi-continuous. Due to (c1), $\Theta_t(x_t)$ is $\mathcal{F}_t$-stable and it is L^0_t-norm bounded (c3a). It follows from Proposition 5.1.5 that there exists $z^*_t \in \Theta_t(x_t)$ such that

$$f_t(x_t,z^*_t) = \max_{z\in\Theta_t(x_t)} u_t(x_t,y_{t+1}(v_t(x_t,z_t)),z_t).$$

The claim follows from the induction hypothesis together with the Bellman principle.

(ii) By (v1), (u1) and (c1), y_t is $\mathcal{F}_t$-stable. Suppose that $\lim_k x_k = x$ a.s. in $L^0_t(H)$ and prove that $\limsup_k y_t(x_k)\le y_t(x)$. To reach a contradiction, suppose that

$$\limsup_k y_t(x_k) > y_t(x) \text{ on } a \in \bar{\mathcal{F}}_t\setminus\{0\}.$$

Arguing on some representative $A\in\mathcal{F}_t$ of a as an autonomous probability space, we can suppose w.l.o.g. that $A=\Omega$. For all $k\in\mathbb{N}$ there exists a maximizer $z^*_k\in\Theta_t(x_k)$ of $f_t(x_k,\cdot)$ due to the previous step. Since f_t is $\mathcal{F}_t$-stable, putting $z^*_{\mathfrak{n}} := \sum_k 1_{\{\mathfrak{n}=k\}} z^*_k$ and $x_{\mathfrak{n}} := \sum_k 1_{\{\mathfrak{n}=k\}} x_k$, $\mathfrak{n}\in L^0_t(\mathbb{N})$, it follows from the $\mathcal{F}_t$-stability of y_t that $z^*_{\mathfrak{n}}\in\Theta_t(x_{\mathfrak{n}})$ is a maximizer of $f_t(x_{\mathfrak{n}}, \cdot)$. For each $k \in \mathbb{N}$, let $M_k := M(x_k)$ be given by (c3). For $\mathfrak{n} \in L_t^0(\mathbb{N})$, put $M_{\mathfrak{n}} := \sum_k 1_{\{\mathfrak{n}=k\}} M_k$. Note that, due to (c1) and (c3a), one has $|z_{\mathfrak{n}}^*| \le M_{\mathfrak{n}}$ for all $\mathfrak{n} \in L_t^0(\mathbb{N})$. By Proposition 5.1.3, there exists a sequence $\mathfrak{n}_1 < \mathfrak{n}_2 < \dots$ in $L_t^0(\mathbb{N})$ such that $\lim_k M_{\mathfrak{n}_k} = \liminf_k M(x_k)$ a.s.. This limit is finite due to (c3b); therefore, $|z_{\mathfrak{n}_k}^*| < \sup_k M_{\mathfrak{n}_k} < \infty$ for all $k \in \mathbb{N}$. Also, since $\limsup_k y_t(x_k) > y_t(x)$, we can assume $(\mathfrak{n}_k)$ constructed so that there exists $\varepsilon \in L_{t,++}^0$ such that

$$y_t(x_{\mathfrak{n}_k}) > y_t(x) + \varepsilon \quad \text{for all } k \in \mathbb{N}. \tag{5.2}$$

Using Proposition 5.1.2, one finds a sequence $\mathfrak{m}_1 < \mathfrak{m}_2 < \dots$ in $L_t^0(\mathbb{N})$ such that $\lim_k z_{\mathfrak{m}_k}^* = z$ for some $z \in L_t^0(\mathbb{R}^n)$. By (c2), we have $z \in \Theta_t(x)$. By the same argument as above, $z_{\mathfrak{m}_k}^*$ is a maximizer of $f_t(x_{\mathfrak{m}_k}, \cdot)$, each $k \in \mathbb{N}$. From (u2), (u3) and the induction hypothesis one has

$$\begin{aligned}
&\limsup_k y_t(x_{\mathfrak{m}_k}) = \limsup_k u_t(x_{\mathfrak{m}_k}, y_{t+1}(v_t(x_{\mathfrak{m}_k}, z_{\mathfrak{m}_k}^*)), z_{\mathfrak{m}_k}^*)\\
&\le \limsup_k u_t(x_{\mathfrak{m}_k}, \sup_{k' \ge k} y_{t+1}(v_t(x_{\mathfrak{m}_{k'}}, z_{\mathfrak{m}_{k'}}^*)), z_{\mathfrak{m}_k}^*)\\
&\le u_t(x, \limsup_k y_{t+1}(v_t(x_{\mathfrak{m}_k}, z_{\mathfrak{m}_k}^*)), z)\\
&\le u_t(x, y_{t+1}(v_t(x, z)), z) = f_t(x, z) \le y_t(x),
\end{aligned}$$

but this contradicts (5.2).

□

We consider the following replacements of the conditions (c2) and (c3b):

(c2') $\Theta_t(x)$ is sequentially closed for any $x \in L_t^0(H)$;

(c3b') For any $x \in L_t^0(H)$ and $n \in \mathbb{N}$ there exists $\delta_n \in L_{t,++}^0$ such that

$$|x - x'| \le \delta_n \quad \text{implies} \quad \Theta_t(x') \subset \Theta_t(x) + B_{\frac{1}{n}}(0).$$

We have the following result:

Theorem 5.2.2. *Suppose that (v1), (v2), (u1)–(u3) and (c1), (c2'), (c3a) and (c3b') are fulfilled. For all $t = 0, \dots, T-1$ the following assertions are valid.*

(i) For all $x_t \in L_t^0(H)$ there exists $((x_s^*)_{s=t}^T, (z_s^*)_{s=t}^{T-1}) \in C_t(x_t)$ such that

$$\begin{aligned}
y_s(x_s) &= u_s(x_s, \cdot, z_s^*) \circ \dots \circ u_{T-1}(x_{T-1}^*, \cdot, z_{T-1}^*) \circ u_T(x_T^*)\\
&= u_s(x_s, y_{s+1}(v_s(x_s, z_s^*)), z_s^*), \quad s = t, \dots, T-1.
\end{aligned}$$

(ii) The function $y_t \colon L_t^0(H) \to \bar{L}_t^0([-\infty, \infty))$ is $\mathcal{F}_t$-stable and sequentially upper semi-continuous.

Proof. We argue again by backward induction. We have that (i)-(ii) are fulfilled at time T by definition. Let us suppose that the assertions are satisfied at time $t+1$, and let us prove (i)-(ii) at time t.

(i) By a copy of (i) of the proof of Theorem 5.2.1, we prove analogously that, for fixed $x_t \in L_t^0(H)$, the function $f_t(x_t, \cdot)$ is sequentially upper semi-continuous. Then, it attains a maximum in $\Theta_t(x_t)$ due to (c1), (c2') and (c3a), as a consequence of Proposition 5.1.5.

(ii) Again we have that y_t is $\mathcal{F}_t$-stable. Suppose $\lim_k x_k = x$ a.s. in $L^0_t(H)$. For any $k \in \mathbb{N}$ there is a maximizer $z^*_k \in \Theta_t(x_k)$. For any $\mathfrak{n} \in L^0_t(\mathbb{N})$ let $z^*_{\mathfrak{n}} := \sum_k 1_{(\mathfrak{n}=k)} z^*_k$, which is an maximizer for $f_t(x_{\mathfrak{n}}, \cdot)$. Due to (c1) and (c3b'), we can find a sequence $\mathfrak{n}_1 < \mathfrak{n}_2 < \dots$ in $L^0_t(\mathbb{N})$ such that $z^*_{\mathfrak{n}_k} \in \Theta_t(x_t) + B_{1/k}(0)$. Thus, put $z^*_{\mathfrak{n}_k} = z_k + u_k$ with $z_k \in \Theta_t(x_t)$ and $u_k \in B_{1/k}(0)$, each $k \in \mathbb{N}$. Due to (c1), (c2') and (c3a), by Proposition 5.1.2, we can find $\mathfrak{m}_1 < \mathfrak{m}_2 < \dots$ in $L^0_t(\mathbb{N})$ so that $\lim_k z_{\mathfrak{m}_k} = z$ a.s. in $\Theta_t(x_t)$. Now, we follow as in (ii) of the proof of Theorem 5.2.1.

□

In the next result, we do not assume constraints on the controls, but derive (c1)–(c3) for super-level sets of y_t as a result of stronger assumptions on the forward and backward generators. For $a \in \mathbb{R} \cup \{-\infty\}$, set $H := (a, +\infty)$. Let

$$v_t : L^0_t(H) \times L^0_t(\mathbb{R}^n) \to L^0_{t+1}(H), \quad t = 0, 1, \dots, T-1$$

be the forward generator satisfying (v1), (v2) and

(v3) increasingness in the first component;

(v4) L^0_t-*concavity*:

$$v_t(x, \eta z + (1-\eta)z') \geq \eta v_t(x, z) + (1-\eta) v_t(x, z')$$

for all $z, z' \in L^0_t(\mathbb{R}^n)$ and $\eta \in L^0_t$ with $0 \leq \eta \leq 1$;

(v5) $\bar{P}_t(v_t(x,z) < x) > 0$ for all $x \in L^0_t(H)$ and $z \in L^0_t(\mathbb{R}^n)$ with $|z| > 0$ where $\bar{P}_t(a) := E_{P_t}[1_a|\mathcal{F}_t]$;

(v6) $v_t(x, 0) = x$ for all $x \in L^0_t(H)$.

As for the backward generator, let $u_T \colon L^0_T(H) \to L^0_T(H)$ be the identity and

$$u_t : L^0_t(H) \times \bar{L}^0_{t+1}([-\infty, \infty)) \times L^0_t(\mathbb{R}^n) \to \bar{L}^0_t([-\infty, \infty)), \quad t = 0, \dots, T-1$$

satisfy (u1), (u3) and

(u2') increasingness in the first and second component;

(u4) L^0_t-*quasi-concavity*:

$$u_t(x, \eta y + (1-\eta)y', \eta z + (1-\eta)z') \geq \inf\{u_t(x, y, z), u_t(x, y', z')\}$$

for all $x \in L^0_t(H)$, $y, y' \in \bar{L}^0_{t+1}([-\infty, \infty))$, $z, z' \in L^0_t(\mathbb{R}^n)$ and $\eta \in L^0_t$ with $0 \leq \eta \leq 1$;

(u5) L^0_t-*cash invariance*: $u_t(x, y+\eta, z) = u_t(x, y, z) + \eta$ for all $x \in L^0_t(H)$, $y \in \bar{L}^0_{t+1}([-\infty, \infty))$, $z \in L^0_t(\mathbb{R}^n)$ and $\eta \in L^0_t$;

(u6) $\mathcal{F}_t$-*sensitivity to large losses*: If $y \in \bar{L}^0_{t+1}([-\infty, \infty))$ is such that $\bar{P}_t(y < 0) > 0$, then $\lim_{r\to\infty} u_t(x, ry, rz) = -\infty$ a.s. for all $z \in L^0_t(\mathbb{R}^n)$;

(u7) $u_t(x, 0, 0) = 0$ for all $x \in L^0_t(H)$.

Similarly as in the setting of Theorem 5.2.1 introduce forward and backward processes. Analogously, we have a Bellman principle. Now we work without control set, and for a given $x_t \in L^0_t(H)$ we define again

$$C_t(x_t) := \left\{((x_s)_{s=t}^T, (z_s)_{s=t}^{T-1}) \colon x_{s+1} = v_s(x_s, z_s),\ z_s \in L^0_t(\mathbb{R}^n) \text{ for } s = t, \dots, T-1\right\}.$$

Theorem 5.2.3. *Suppose that (v1)–(v6), (u1), (u2'), and (u3)–(u7) are fulfilled. If for every $t=0,\dots,T-1$ and $x\in L^0_t(H)$ there is some $M_t(x)\in L^0_t$ such that*

$$y_{t+1}(v_t(x,z))-v_t(x,z)\le M_t(x)\quad \text{for all } x\in L^0_t(H),\ z\in L^0_t(\mathbb{R}^n), \tag{5.3}$$

then

(i) for all $t=0,\dots,T-1$ and $x_t\in L^0_t(H)$ there exists $((x^*_s)_{s=t}^T,(z^*_s)_{s=t}^{T-1})\in C_t(x_t)$ such that

$$\begin{aligned} y_s(x_s)&=u_s(x_s,\cdot,z^*_s)\circ\ldots\circ u_{T-1}(x^*_{T-1},\cdot,z^*_{T-1})\circ u_T(x^*_T)\\ &=u_s(x_s,y_{s+1}(v_s(x_s,z^*_s)),z^*_s),\quad s=t,\dots,T-1.\end{aligned}$$

(ii) $y_t\colon L^0_t(H)\to L^0_t$ is $\mathcal{F}_t$-stable, increasing and sequentially upper semi-continuous for all $t=0,\dots,T$.

Proof. First, we prove

$$x\le y_t(x) \tag{5.4}$$

for all $x\in L^0_t(H)$ and $t=0,\dots,T-1$. By induction, suppose that $x\le y_{t+1}(x)$. Then

$$y_t(x)\ge u_t(x,y_{t+1}(v_t(x,0)),0)\ge x$$

where the second inequality follows from (v6), the induction hypothesis and (u2'), (u5) and (u7).

(i) For $t=0,\dots,T-1$ and $x\in L^0_t(H)$, if

$$f_t(x,\cdot)\colon L^0_t(\mathbb{R}^n)\longrightarrow \bar L^0_t([-\infty,\infty)),\quad f_t(x,z):=u_t(x,y_{t+1}(v_t(x,z)),z)$$

has a maximizer, then it lies in

$$\Theta_t(x):=\{z\in L^0_t(\mathbb{R}^n)\colon f_t(x,z)\ge f_t(x,0)=u_t(x,y_{t+1}(x),0)\}.$$

By inspection, it follows from (v2) and (u3) that $f_t(x,\cdot)$ is sequentially upper semi-continuous and $\Theta_t(x)$ is sequentially closed. Also, $\Theta_t(x)$ satisfies (c1) due to (v1) and (u1). By arguing similarly as in Theorem 5.2.1, it is enough to prove that $\Theta_t(x)$ is L^0_t-norm bounded. Let $z\in L^0_t(\mathbb{R}^n)$. By assumption (5.3), one has

$$y_{t+1}(v_t(x,z))\le v_t(x,z)+M_t(x).$$

Applying (u2') and (u6) to the previous inequality we obtain

$$f_t(x,z)\le u_t(x,v_t(x,z),z)+M_t(x). \tag{5.5}$$

Hence it is enough to prove that $\Xi_t(x)$ is L^0_t-norm bounded where

$$\Theta_t(x)\subset\Xi_t(x):=\left\{z\in L^0_t(\mathbb{R}^n)\colon u_t(x,v_t(x,z),z)\ge f_t(x,0)-M_t(x)\right\}.$$

By (v4) and (u4), one has that $\Xi_t(x)$ is L^0_t-convex. The L^0_t-norm boundedness of $\Xi_t(x)$ would follow from Lemma 5.1.1, if for all $z\in\Xi_t(x)$ with $z\neq 0$, there exists $a\in\mathcal{A}_t\setminus\{0\}$ such that

$$\lim_{m\to\infty}u_t(x,v_t(x,mz),mz)=-\infty,\quad \text{on } a \tag{5.6}$$

since $f_t(x,0)\ge x$ by applying (v6), (u5) and (u7) to (5.4).

Fix $z_0 \in \Xi_t(x) \setminus \{0\}$. Take $z \in L^0_t(\mathbb{R}^n)$ with $|z| > 0$ and $z = z_0$ on $\operatorname{supp}(z_0)$. First, we will prove that there is $\mathfrak{n} \in L^0_t(\mathbb{N})$ with

$$\bar{P}_t\left(x^+ + \mathfrak{n}(v_t(x,z) - x) < 0\right) > 0. \tag{5.7}$$

Indeed, by (v5) we know that $\bar{P}_t(e) > 0$ for $e := \{v_t(x,z) < x\} \in \bar{\mathcal{F}}_{t+1}$. Now we can find a partition $(e_k) \in p_{t+1}(e)$ with $\bigvee_k e_k = e$ and $(n_k) \subset \mathbb{N}$ such that

$$x^+ + n_k(v_t(x,z) - x) < 0 \quad \text{on } e_k \quad \text{for each } k \in \mathbb{N}.$$

Then,

$$0 < \bar{P}_t(e) = \sum \bar{P}_t(e_k) \leq \sum \bar{P}_t(x^+ + n_k(v_t(x,z) - x) < 0).$$

Let $b_k := \{\bar{P}_t(x^+ + n_k(v_t(x,z) - x) < 0) > 0\} \in \bar{\mathcal{F}}_t$, and put $a_1 := b_1$ and $a_k := b_k \wedge (\bigvee_{k' < k} b^c_{k'})$, $k \geq 2$. We have that $(a_k) \in p_t(I)$. Then $\mathfrak{n} := \sum_k 1_{a_k} n_k$ yields the desired.

Second, by (v4) and (v6) one has

$$v_t(x, mz) - x \leq m\,(v_t(x,z) - x) \quad \text{for all } m \in \mathbb{N}. \tag{5.8}$$

Let $a \in \bar{\mathcal{F}}_t \setminus \{0\}$ be so that $a \leq \operatorname{supp}(z_0)$ and $\mathfrak{n} = k$ on a for some $k \in \mathbb{N}$. It follows from (5.8) by choosing $m \in \mathbb{N}$ large enough

$$\begin{aligned} u_t(x, v_t(x,mz), mz) &\leq u_t(x, x^+ + v_t(x,mz) - x, mz) \\ &\leq u_t\left(x, \frac{m}{k}\left(x^+ + k(v_t(x,z) - x)\right), mz\right), \quad \text{on } a. \end{aligned}$$

We obtain (5.6) from (u6) and (5.7) by letting m tend to infinity and by just noting that $z = z_0$ on a.

(ii) Due to (v3) and (u2'), y_t is increasing. In view of the last part of Proposition 5.1.4, it is enough to show that if $x_k \searrow x$ a.s. in $L^0_t(H)$, then $y_t(x_k) \searrow y_t(x_0)$ a.s..

By monotonicity, one has

$$f_t(x_1, z) \geq f_t(x_2, z) \geq \ldots \geq f_t(x_0, z), \quad \text{for all } z \in L^0_t(\mathbb{R}^n). \tag{5.9}$$

Now, for each $k \in \mathbb{N} \cup \{0\}$ define $\bar{\Theta}_t(x_k) := \{z \in L^0_t(\mathbb{R}^n) \colon f_t(x_k, z) \geq f_t(x_0, 0)\}$. Due to (5.9), we have

$$\bar{\Theta}_t(x_0) \subset \ldots \subset \bar{\Theta}_t(x_{k+1}) \subset \bar{\Theta}_t(x_k) \subset \ldots \subset \bar{\Theta}_t(x_1).$$

Further, the sets above are non-empty as $0 \in \Theta_t(x_0)$, and the maximizer z^*_k of $f_t(x_k, z)$ lies in $\bar{\Theta}_t(x_k)$ for each $k \in \mathbb{N} \cup \{0\}$. Besides, due to (5.5) and (5.9), we have

$$\bar{\Theta}_t(x_k) \subset \bar{\Xi}_t(x_1) := \left\{z \in L^0_t(\mathbb{R}^n) \colon u_t(x_1, v_t(x_1, z), z) \geq f_t(x_0, 0) - M_t(x_1)\right\}$$

for all $k \in \mathbb{N} \cup \{0\}$. By arguing as in (i), we can prove that $\bar{\Xi}_t(x_1)$ is L^0_t-norm bounded. Then, (z^*_k) is L^0_t-norm bounded, and due to Proposition 5.1.2 we find $\mathfrak{n}_1 < \mathfrak{n}_2 < \ldots$ in $L^0_t(\mathbb{N})$, such that $\lim_k z^*_{\mathfrak{n}_k} = z^*$ a.s..

Now, we have

$$y_t(x) \leq \lim_k y_t(x_{\mathfrak{n}_k}) = \lim_k u_t(z^*_{\mathfrak{n}_k}, y_{t+1}(v_t(x_{\mathfrak{n}_k}, z^*_{\mathfrak{n}_k})), z^*_{\mathfrak{n}_k}) \leq$$

$$\leq u_t(z^*, y_{t+1}(v_t(x, z^*)), z^*) \leq y_t(x).$$

□

Remark 5.2.1. *If for all $t = 0, 1, \ldots, T-1$ there exists a constant $M > 0$ such that*

$$\sup_z u_t(x, v_t(x,z), z) - x \le M \quad \text{for all } x \in L^0_t(H),$$

then condition (5.3) is satisfied for $M_t(x) = (T - t + 1)M$. Indeed, by induction suppose that $y_{t+1}(x) - x \le (T-t)M$. Then

$$\begin{aligned} y_t(x) - x &\le u_t(x, y_{t+1}(v_t(x,z)), z) - x \\ &= u_t(x, y_{t+1}(v_t(x,z)) - v_t(x,z) + v_t(x,z), z) - x \\ &\le u_t(x, v_t(x,z), z) - x + (T-t)M \\ &\le (T - t + 1)M \end{aligned}$$

where in the second inequality we used (u2') and (u5).

5.3 Related examples

In this section, we will see some related mathematical finance problems. Throughout, we will suppose that our filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t=0}^T, P)$ models the flow of available market information through the time.

5.3.1 Wealth-dependent utility maximization under short-selling constraints

Let $(S_t)_{t=0}^T$ be an adapted $(0,\infty)^n$-valued stochastic process representing a discounted market model with n securities. Denote by $\Delta S_t := S_t - S_{t-1}$, $t = 1, \ldots, T$, the price increment process. Recall that an $\mathbb{R}^n$-valued predictable process $(\vartheta_t)_{t=1}^T$ is called a self-financing trading strategy if

$$\langle \vartheta_t, S_t \rangle = \langle \vartheta_{t+1}, S_t \rangle, \quad t = 1, \ldots, T-1. \tag{5.10}$$

Given a self-financing trading strategy $(\vartheta_t)_{t=1}^T$, define a value process

$$\begin{aligned} x_0 &= \langle \vartheta_1, S_0 \rangle, \\ x_{t+1} &:= x_t + \langle \vartheta_{t+1}, \Delta S_{t+1} \rangle \quad t = 0, \ldots, T-1. \end{aligned}$$

Assume the short-selling constraint

$$-b_t^i \le \vartheta_t^i, \quad t = 1, \ldots, T, \ i = 1, \ldots, n \tag{5.11}$$

for a $[0,\infty)^n$-valued predictable process $(b_t)_{t=1}^T$. For $t = 0, \ldots, T-1$ and $x \in L^0_{t,+}$, define

$$\Theta_t(x) := \left\{ \vartheta \in L^0_t(\mathbb{R}^n) \colon x = \langle \vartheta, S_t \rangle, \ -b^i_{t+1} \le \vartheta^i, i = 1, \ldots, n \right\}.$$

For $i = 1, \ldots, n$, put

$$M_t^i(x) := \frac{1}{S_t^i} \left(x + \sum_{j=1, j \ne i}^n -b^j_{t+1} S^j_t \right).$$

Set $M_t(x) := \max\{M_t^i(x) \colon i = 1, \ldots, n\}$. By inspection, $x \mapsto \Theta_t(x)$ fulfills (c1)–(c3). Since the forward generator $v_t(x, \vartheta) := x + \langle \vartheta, \Delta S_{t+1} \rangle$, $t = 0, \ldots, T-1$, satisfies (v1) and (v2), we obtain for any initial wealth $x_0 \in L^0_0$ the existence of an optimal trading strategy $(\vartheta_t^*)_{t=1}^T$ in $\Theta_0(x_0) \times \Theta_1(x_1) \times \ldots \times \Theta_{T-1}(x_{T-1})$ such that

$$y_0(x_0) = u_0(x_0, \cdot, \vartheta_1^*) \circ \ldots \circ u_{T-1}(x_{T-1}, \cdot, \vartheta_T^*) \circ u_T(x_T)$$

for any dynamic wealth-dependent utility $(u_t)_{t=0}^T$ satisfying (u1)–(u3).

5.3.2 Wealth-dependent utility maximization of generalized entropic utilities

Let $(S_t)_{t=0}^T$ be an adapted $\mathbb{R}^n$-valued stochastic process representing a discounted market model. Denote by $\Delta S_t := S_t - S_{t-1}$, $t = 1, \dots, T$, the price increment process. For an $\mathbb{R}^n$-valued predictable process $(\vartheta_t)_{t=1}^T$ and initial wealth $x_0 \in L_0^0$, define recursively a value process

$$x_{t+1} = v_t(x_t, \vartheta_{t+1}) := x_t + \langle \vartheta_{t+1}, \Delta S_{t+1} \rangle, \quad t = 0, \dots, T-1.$$

By inspection, v_t satisfies (v1)–(v6) for all $t = 0, \dots, T-1$ under the no-arbitrage assumption

$$x_0 = 0 \text{ and } x_T \geq 0 \quad \text{implies} \quad x_T = 0.$$

Let $t = 0, \dots, T-1$ and $\gamma_t \colon L_t^0 \to L_{t,++}^0$ be $\mathcal{F}_t$-stable, decreasing and sequentially continuous, modelling a wealth-dependent risk aversion coefficient at time t. Let further

$$G_t \colon \bar{L}_{t+1}^0([-\infty, \infty)) \to \bar{L}_t^0([-\infty, \infty))$$

be increasing, L_t^0-concave, $\mathcal{F}_t$-translation invariant, sequentially upper semi-continuous with $G_t(0) = 0$ and $\lim_r G_t(ry) = -\infty$ whenever $P_t(y < 0) > 0$. For instance, one could think of the entropic preference functional $G_t(y) = -\log E_P[\exp(-y)|\mathcal{F}_t]$. Let $u_T \colon L_T^0 \to L_T^0$ be the identity and

$$u_t : L_t^0 \times \bar{L}_{t+1}^0([-\infty, \infty)) \to \bar{L}_t^0([-\infty, \infty)), \quad u_t(x, y) := \frac{1}{\gamma_t(x)} G_t(\gamma_t(x) y), \quad t = 0, \dots, T-1.$$

Let us check that u_t satisfies (u1),(u2'),(u3)–(u7) for all $t = 0, \dots, T$:

By using that G_t is L_t^0-concave it can be shown that it is $\mathcal{F}_t$-stable (u1). Let us show (u2'). Indeed, let us take $x_1 \leq x_2$ and $y_1 \leq y_2$. For $x \in L_t^0$, we have

$$G_t(\frac{y_2}{\gamma_t(x_2)}) \geq G_t(\frac{y_1}{\gamma_t(x_2)}) \geq \frac{\gamma_t(x_1)}{\gamma_t(x_2)} G_t(\frac{y_1}{\gamma_t(x_1)}) + \frac{\gamma_t(x_2) - \gamma_t(x_1)}{\gamma_t(x_2)} G_t(0)$$
$$= \frac{\gamma_t(x_1)}{\gamma_t(x_2)} G_t(\frac{y_1}{\gamma_t(x_1)}).$$

Multiplying by $\gamma_t(x_2)$ we obtain

$$u_t(x_2, y_2) = \gamma_t(x_2) G_t(\frac{y_2}{\gamma_t(x_2)}) \geq \gamma_t(x_1) G_t(\frac{y_1}{\gamma_t(x_1)}) = u_t(x_1, y_1).$$

Let us show (u3). By the last part of Proposition 5.1.4, it is enough to check that $x_k \searrow x$ a.s. and $y_k \searrow y$ a.s. implies $u_t(x_k, y_k) \searrow u_t(x, y)$ a.s.. Indeed, if $x_k \searrow x$ a.s. and $y_k \searrow y$ a.s., then, by monotonicity of u_t we have

$$G_t(\frac{y_k}{\gamma_t(x_k)}) \geq \frac{\gamma_t(x)}{\gamma_t(x_k)} G_t(\frac{y}{\gamma_t(x)}) \quad \text{for all } k \in \mathbb{N}.$$

By taking limsup above and by using that γ_t is sequentially continuous and G_t is sequentially upper semi-continuous, we have

$$G_t(\frac{y}{\gamma_t(x)}) \geq \limsup_k G_t(\frac{y_k}{\gamma_t(x_k)}) \geq \liminf_k G_t(\frac{y_k}{\gamma_t(x_k)})$$
$$\geq \lim_k \frac{\gamma_t(x)}{\gamma_t(x_k)} G_t(\frac{y}{\gamma_t(x)}) = G_t(\frac{y}{\gamma_t(x)}).$$

We finally obtain $\lim_k u_t(x_k, y_k) = u_t(x, y)$ a.s..

Since G_t is L^0_t-concave, it follows that $u_t(x, \cdot)$ is L^0_t-concave. In particular, (u4) follows.

It is simply to verify (u5), (u6) and (u7).

For a value process $(x_t)_{t=0}^T$, define the backward process

$$y_t(x_t) = \sup_{((x_s)_{s=t}^T, (\vartheta_s)_{s=t}^{T-1}) \in C_t(x_t)} u_t(x_t, \cdot) \circ \ldots \circ u_{T-1}(x_{T-1}, \cdot) \circ u_T(x_T), \quad t = 0, \ldots, T-1$$

where $C_t(x_t) := \Big\{((x_s)_{s=t}^T, (\vartheta_s)_{s=t}^{T-1})\colon x_{s+1} = x_s + \langle \vartheta_{s+1}, \Delta S_{s+1}\rangle, \text{ for } s = t, \ldots, T-1\Big\}$.

By induction, one can verify that $y_t(x + \eta) = y_t(x) + \eta$ for any $\eta \in L^0_{t-1}$ with $t = 1, \ldots, T$. Then, if for every $t = 1, \ldots, T$ one can find $M_{t-1} \in L^0_{t-1}$ such that

$$y_t(\vartheta \cdot \Delta S_t) - \vartheta \cdot \Delta S_t \le M_{t-1} \quad \text{for all } \vartheta \in L^0_{t-1}(\mathbb{R}^n), \tag{5.12}$$

then it follows from Theorem 5.2.3 that

$$y_0(x_0) = \sup_{((x_t)_{t=0}^T, (\vartheta_t)_{t=0}^{T-1}) \in C_0(x_0)} u_0(x_0, \cdot) \circ \ldots \circ u_{T-1}(x_{T-1}, \cdot) \circ u_T(x_T)$$

is attained for any initial wealth x_0.

5.3.3 Wealth-dependent dynamic utility sharing

Suppose that we have a finite number n of agents that operate in a financial market. Each agent $i = 1, \ldots, n$ is endowed with a wealth process $(H^i_t)_{t=0}^T$ with $H^i_t \in L^0_{t,++}$. The aim is to share optimally the aggregated endowment process $H_t = \sum_{i=1}^n H^i_t$, $t = 0, \ldots, T$, with respect to a dynamic wealth-dependent utility. The utilities under consideration are of the form

$$u_t\colon L^0_{t,++} \times L^0_{t+1,++} \to \bar{L}^0_t([-\infty, \infty)), \quad u_t(x, y) := xG_t(y/x)$$

where $G_t : \bar{L}^0_{t+1}([-\infty, \infty)) \to \bar{L}^0_t([-\infty, \infty))$ is an L^0_t-concave, increasing and sequentially upper semi-continuous generator with $G_t(0) = 0$ for all $t = 0, \ldots, T-1$, and $u_T\colon L^0_{T,++} \to \bar{L}^0_t([-\infty, \infty))$ is the identity.

Proposition 5.3.1. *The wealth-dependent dynamic optimal utility sharing problem*

$$y_t((H^i_t)_{i=1}^n) = \sup\left\{\sum_{i=1}^n u_t(H^i_t, \cdot) \circ u_{t+1}(x^i_{t+1}, \cdot) \circ \ldots \circ u_{T-1}(x^i_{T-1}, x^i_T)\colon \begin{array}{c} \sum_{i=1}^n x^i_s = H_s, \\ x^i_s \in L^0_{s,++},\ s = t+1, \ldots, T \end{array}\right\}$$

has the optimal solution

$$x^{i,*}_s = \frac{H^i_t}{H_t} H_s, \quad i = 1, \ldots, n,\ s = t+1, \ldots, T.$$

Moreover, the function $y_t\colon L^0_t((0, \infty)^n) \to \bar{L}^0_t([-\infty, \infty))$ is $\mathcal{F}_t$-stable, increasing and sequentially upper semi-continuous for $t = 0, \ldots, T$.

Proof. Define

$$\bar{y}_T\colon L^0_T((0, \infty)^n) \longrightarrow L^0_T, \quad \bar{y}_T((x^i)_{i=1}^n) = \sum_{i=1}^n x^i,$$

and for $t = 0, \dots, T-1$, let

$$\bar{y}_t \colon L^0_t\left((0,\infty)^n\right) \longrightarrow \bar{L}^0_t([-\infty,\infty)), \quad \bar{y}_t\left(\left(x^i\right)_{i=1}^n\right) = \left(\sum_{i=1}^n x^i\right) G_t\left(\frac{\bar{y}_{t+1}\left((H^i_{t+1})_{i=1}^n\right)}{\sum_{i=1}^n x^i}\right).$$

We show that

$$\sum_{i=1}^n u_t(H^i_t,\cdot) \circ u_{t+1}(x^{i,*}_{t+1},\cdot) \circ \dots \circ u_{T-1}(x^{i,*}_{T-1}, x^{i,*}_T) = \bar{y}_t((H^i_t)_{i=1}^n) \tag{5.13}$$

and

$$\sum_{i=1}^n u_t(H^i_t,\cdot) \circ u_{t+1}(x^{i}_{t+1},\cdot) \circ \dots \circ u_{T-1}(x^{i}_{T-1}, x^{i}_T) \le \bar{y}_t((H^i_t)_{i=1}^n) \tag{5.14}$$

for all $(x^i_s)_{i=1}^n$ such that $\sum_{i=1} x^i_s = H_s$, $s = t+1, \dots, T$. It would follow from (5.13) and (5.14) that

$$x^{i,*}_s = \frac{H^i_t}{H_t} H_s, \quad i = 1, \dots, n,\ s = t+1, \dots, T$$

is an optimal solution and that $y_t((H^i_t)_{i=1}^n) = \bar{y}_t((H^i_t)_{i=1}^n)$. By backward induction, it can be checked that

$$u_s(x^{i,*}_s,\cdot) \circ u_{s+1}(x^{i,*}_{s+1},\cdot) \circ \dots \circ u_{T-1}(x^{i,*}_{T-1}, x^{i,*}_T) = x^{i,*}_s \frac{\bar{y}_s((H^i_s)_{i=1}^n)}{H_s} \tag{5.15}$$

for all $i = 1, \dots, n$ and $s = t, \dots, T$ where we put $x^{i,*}_t = H^i_t$. By summing up (5.15) at $s = t$ over $i = 1, \dots, n$ we obtain (5.13).

We prove (5.14) by backward induction. It is true at T by definition. Let $t \le s < T$. Then

$$\begin{aligned}
&\sum_{i=1}^n u_s(x^i_s,\cdot) \circ u_{s+1}(x^i_{s+1},\cdot) \circ \dots \circ u_{T-1}(x^i_{T-1}, x^i_T) \\
&= H_s \sum_{i=1}^n \frac{x^i_s}{H_s} G_s\left(\frac{1}{x^i_s} u_{s+1}(x^i_{s+1},\cdot) \circ \dots \circ u_{T-1}(x^i_{T-1}, x^i_T)\right) \\
&\le H_s G_s\left(\sum_{i=1}^n \frac{x^i_s}{H_s}\frac{1}{x^i_s} u_{s+1}(x^i_{s+1},\cdot) \circ \dots \circ u_{T-1}(x^i_{T-1}, x^i_T)\right) \\
&\le H_s G_s\left(\frac{y_{s+1}((H^i_{s+1})_{i=1}^n)}{H_s}\right) = y_s((H^i_s)_{i=1}^n)
\end{aligned}$$

where the first inequality follows from L^0_s-concavity and the last one by monotonicity of G_s and the induction hypothesis.

Finally, for any $t = 0, \dots, T$, we have that $y_t((x^i)_{i=1}^n) = \bar{y}_t((x^i)_{i=1}^n)$ for any $(x^i)_{i=1}^n \in L^0_t((0,\infty)^n)$. By using the recursive definition of $\bar{y}_t$ it can be proven by backward induction that $\bar{y}_t$ (and thus y_t) is $\mathcal{F}_t$-stable, increasing and sequentially upper semi-continuous for $t = 0, \dots, T$. □

5.4 Measurable closed-valued mappings, measurable compact-valued mappings and stable sequentially closed subsets of $L^0(\mathbb{R}^d)$

In this section we will deal with representatives of elements of $L^0(\mathbb{R})$. In general, if $\eta \in L^0(\mathbb{R})$, we denote by η^0 some representative of η, which is arbitrarily chosen unless we say otherwise.

Throughout this section $F:\Omega\rightrightarrows\mathbb{R}^d$ denotes a set-valued mapping. Let us introduce some terminology. We define the *effective domain* of F to be $\mathrm{dom}(F):=\{\omega\in\Omega\colon F(\omega)\neq\emptyset\}$. We will say that F is *measurable* whenever its graph $\{(\omega,x)\in\Omega\times\mathbb{R}^d\colon x\in F(\omega)\}$ is an element of the σ-algebra $\mathcal{F}\otimes\mathcal{B}(\mathbb{R}^d)$. We will say that F is *closed-valued* (resp. *compact-valued*) if $F(\omega)$ is a closed (resp. compact) subset of $\mathbb{R}^d$ for almost every $\omega\in\Omega$. A *measurable selection*, or simply selection, of F is an element $\eta\in L^0(\mathbb{R}^d)$ such that $\eta^0(\omega)\in S(\omega)$ for almost every $\omega\in\Omega$, where η^0 is an arbitrary representative of η. We denote by X_F the set of all selections of F.

Let us see some well-known results within the theory of measurable selections:

Lemma 5.4.1. *[1, Theorem 18.13] (Kuratowski-Ryll-Nardzewski Selection Theorem) Any measurable closed-valued mapping $F:\Omega\rightrightarrows\mathbb{R}^d$ with $\mathrm{dom}(F)=\Omega$ admits a selection.*

Lemma 5.4.2. *[1, Corollary 18.14] (Castaing's representation Theorem) For a closed-valued mapping $F:\Omega\rightrightarrows\mathbb{R}^d$ with $\mathrm{dom}(F)=\Omega$, the following conditions are equivalent:*

1. *There exists a sequence (η_n) in X_F such that $F(\omega)=\overline{\{\eta_1^0(\omega),\eta_2^0(\omega),\ldots\}}$ for all $\omega\in\Omega$, where $\eta_1^0,\eta_2^0,\ldots$ are representatives of $\eta_1,\eta_2,\ldots$, respectively;*

2. *F is measurable.*

Lemma 5.4.3. *[1, Theorem 18.19] (Measurable Maximum Theorem) For a compact-valued mapping $F:\Omega\rightrightarrows\mathbb{R}^d$ with $\mathrm{dom}(F)=\Omega$, suppose that $\lambda:\mathbb{R}^d\to\mathbb{R}$ is continuous, then the function*

$$\eta^0:\Omega\longrightarrow\mathbb{R}\quad \eta^0(\omega):=\max_{x\in F(\omega)}\lambda(x),$$

is $\mathcal{F}$-measurable.

The following result connects measurable closed-valued mappings and closed subsets of $L^0(\mathbb{R}^d)$.

Theorem 5.4.1. *Let S be a subset of $L^0(\mathbb{R}^d)$. The following conditions are equivalent:*

1. *$S=X_F$ for some measurable closed-valued mapping $F:\Omega\rightrightarrows\mathbb{R}^d$ with $\mathrm{dom}(F)=\Omega$;*

2. *S is stable and sequentially closed;*

3. *S is stable and $|\cdot|$-closed.*

In that case, if F,F' are two measurable set-valued mappings satisfying 1, then $F(\omega)=F'(\omega)$ for almost every $\omega\in\Omega$.

Proof. $1\Rightarrow 3$: Suppose that $F:\Omega\rightrightarrows\mathbb{R}^d$ is a closed-valued mapping with $\mathrm{dom}(F)=\Omega$ such that

$$S=\{\eta\in L^0(\mathbb{R}^d)\colon \eta^0(\omega)\in F(\omega)\text{ for a.e. }\omega\},$$

where η^0 is any representative of η. First, due to the Lemma 5.4.1, S is non-empty. It follows by inspection, that S is stable.

Now, take $\eta\in\overline{S}$. For each $n\in\mathbb{N}$, choose $\eta_n\in S$ such that $|\eta-\eta_n|\leq\frac{1}{n}$. Fix representatives η^0 of η and η_n^0 of and η_n, each n. Let

$$A_n:=\{\omega\in\Omega\colon \eta_n^0(\omega)\notin F(\omega),\ |\eta^0(\omega)-\eta_n^0(\omega)|>\frac{1}{n}\},\quad n\in\mathbb{N}.$$

Since A_n has probability 0, we have that $A:=\bigcup_{n\in\mathbb{N}}A_n$ has also probability 0. It follows that $\eta^0(\omega)\in\overline{F(\omega)}=F(\omega)$ for all $\omega\in\Omega\setminus A$, which implies that $\eta\in S$. We conclude that S is $|\cdot|$-closed.

$3 \Rightarrow 1$: Enumerate $\mathbb{Q}^d := \{q_1, q_2, \ldots\}$. For each $m \in \mathbb{N}$, let

$$\delta_m := \inf\{|q_m - \eta| \colon \eta \in S\}.$$

For any $m, n \in \mathbb{N}$, pick $\eta_{n,m} \in S$ such that

$$|q_m - \eta_{n,m}| \le \delta_m + \frac{1}{3n}, \tag{5.16}$$

which is possible since S is stable. For each $n, m \in \mathbb{N}$, choose a representative $\eta^0_{n,m}$ of $\eta_{n,m}$. Define

$$F : \Omega \rightrightarrows \mathbb{R}^d \quad \omega \mapsto \overline{\{\eta^0_{n,m}(\omega) \colon n, m \in \mathbb{N}\}},$$

where, as usual, η is represented by η_0. By Lemma 5.4.2, F is measurable.

We claim that $X_F = S$. Take $\xi \in X_F$. To reach a contradiction, suppose that $\xi \notin S$.

Since S is $|\cdot|$-closed and stable, we have that $S = \{\nu \in L^0(\mathbb{R}^d) \colon \inf_{\eta \in S} |\nu - \eta| = 0\}$. Thus we can find $\varepsilon \in L^0_{++}$ and $a \in \mathcal{A} \setminus \{0\}$ such that

$$|\xi - \eta| > \varepsilon \text{ on } a \quad \text{for all } \eta \in S.$$

In particular, we have that

$$|\xi - \eta_{n,m}| > \varepsilon \text{ on } a \quad \text{for all } n, m \in \mathbb{N}.$$

Fix representatives A, ξ^0 and ε^0 of a, ξ and ε, respectively, with $\varepsilon^0(\omega) > 0$ for all ω. For each $n, m \in \mathbb{N}$, set $A_{n,m} := \{\omega \in A \colon |\xi^0(\omega) - \eta^0_{n,m}(\omega)| \le \varepsilon^0(\omega)\}$. Each $A_{n,m}$ has probability 0, thus $B := \bigcup_{n,m} A_{n,m}$ has probability 0. Then $|\xi^0(\omega) - \eta^0_{n,m}(\omega)| > \varepsilon^0(\omega)$ for all $\omega \in A \setminus B$ and $n, m \in \mathbb{N}$, which contradicts that $\xi^0(\omega) \in \overline{\{\eta^0_{n,m}(\omega) \colon n, m \in \mathbb{N}\}}$ for almost every ω.

To prove the reverse inclusion, take $\xi \in S$. For each $\mathfrak{m} \in L^0(\mathbb{N})$ and $n \in \mathbb{N}$, define

$$q_{\mathfrak{m}} := \sum_k 1_{\{\mathfrak{m}=k\}} q_k, \quad \delta_{\mathfrak{m}} := \sum_k 1_{\{\mathfrak{m}=k\}} \delta_k \quad \text{and} \quad \eta_{n,\mathfrak{m}} := \sum_k 1_{\{\mathfrak{m}=k\}} \eta_{n,k}.$$

Since $\mathbb{Q}^d = \{q_1, q_2, \ldots\}$, we have that $L^0(\mathbb{Q}^d) = \{q_{\mathfrak{m}} \colon \mathfrak{m} \in L^0(\mathbb{N})\}$, where $q_{\mathfrak{m}} = \sum_k 1_{\{\mathfrak{m}=k\}} q_k$ for $\mathfrak{m} \in L^0(\mathbb{N})$. Since $L^0(\mathbb{Q}^d)$ is dense in $L^0(\mathbb{R}^d)$, for any $n \in \mathbb{N}$ we can take $\mathfrak{m}_n \in L^0(\mathbb{N})$ such that $|\xi - q_{\mathfrak{m}_n}| \le 1/3n$. Then, due to (5.16), we have

$$|q_{\mathfrak{m}_n} - \eta_{n,\mathfrak{m}_n}| \le \delta_{\mathfrak{m}_n} + \frac{1}{3n}.$$

Since $\delta_{\mathfrak{m}_n} \le |q_{\mathfrak{m}_n} - \xi| \le \frac{1}{3n}$, we obtain

$$|\xi - \eta_{n,\mathfrak{m}_n}| \le |\eta - q_{\mathfrak{m}_n}| + |q_{\mathfrak{m}_n} - \eta_{n,\mathfrak{m}}| \le \frac{1}{3n} + \frac{1}{3n} + \frac{1}{3n} \le \frac{1}{n} \quad \text{for all } n \in \mathbb{N}.$$

Choose some representative ξ^0 of ξ and, for any $n \in \mathbb{N}$, fix a representative $\eta^0_{n,\mathfrak{m}_n}$ of $\eta_{n,\mathfrak{m}_n}$. Then, for each $n \in \mathbb{N}$, the set

$$A_n := \{\omega \colon |\xi^0(\omega) - \eta^0_{n,\mathfrak{m}_n}(\omega)| \le \frac{1}{n}\} \cap \{\omega \colon \exists m \in \mathbb{N},\ \eta^0_{n,\mathfrak{m}_n}(\omega) = \eta^0_{n,m}(\omega)\}$$

has probability 1. Set $B := \bigcap_n A_n$, which has probability 1. Then $\xi^0(\omega) \in \overline{\{\eta^0_{n,m}(\omega) \colon n, m \in \mathbb{N}\}}$ for all $\omega \in B$. Hence $\xi \in X_F$.

$2 \Leftrightarrow 3$ follows from Corollary 5.1.1

Finally, let us prove that, if $S = X_F = X_{F'}$, then $F(\omega) = F'(\omega)$ for a.e. ω. Indeed, since F and F' are measurable, we have that the sections

$$A_x := \{\omega \in \Omega \colon x \in F(\omega)\} \quad \text{and} \quad A'_x := \{\omega \in \Omega \colon x \in F'(\omega)\}$$

are measurable for all $x \in \mathbb{R}^d$. It suffices to prove that $P(A_x \triangle A'_x) = 0$ for all $x \in R^d$. To reach a contradiction, suppose that, for instance, $P(A_x \setminus A'_x) > 0$. Let $B := A_x \setminus A'_x$ and fix some $\nu \in S$. Since S is stable, $\eta := 1_b x + 1_{b^c}\nu \in S$. Thus η is a selection of F', hence $\eta^0(\omega) \in F'(\omega)$ for a.e. $\omega \in B$, for an arbitrary representative η^0 of η, but this means that $x \in F'(\omega)$ for a.e. $\omega \in B = A_x \setminus A'_x$, which is a contradiction as $P(B) > 0$. □

Notice that, in view of the theorem above, the correspondence $F \mapsto X_F$ defines a bijection between stable closed subsets of $L^0(\mathbb{R}^d)$ and measurable closed-valued mapping $F : \Omega \rightrightarrows \mathbb{R}^d$ with $\mathrm{dom}(F) = \Omega$ identified whenever they coincide on a.e. ω.

Regarding stable compactness, we have the following:

Theorem 5.4.2. *Let S be a subset of $L^0(\mathbb{R}^d)$.*

1. *$S = X_F$ for some compact-valued mapping $F : \Omega \rightrightarrows \mathbb{R}^d$ with dom$(F) = \Omega$;*
2. *S is stable, sequentially closed and $\sup_{\eta\in S} |\eta| < \infty$;*
3. *S is stable, $|\cdot|$-closed and $\sup_{\eta\in S} |\eta| < \infty$;*

In this case, if F, F' are two set-valued mappings satisfying 1, then $F(\omega) = F'(\omega)$ for a.e. ω.

Proof. By the module analogue of the Heine-Borel theorem (see Theorem 2.4.5), we have that S is stably compact if and only if S is $|\cdot|$-closed and L^0-norm bounded. Then the equivalence $2 \Leftrightarrow 3$ follows from Theorem 5.4.1.

$1 \Rightarrow 3$: Suppose that $F : \Omega \rightrightarrows \mathbb{R}^d$ is a compact-valued mapping $F : \Omega \rightrightarrows \mathbb{R}^d$ with $\mathrm{dom}(F) = \Omega$ and $S = X_F$. By Theorem 5.4.1, S is closed. Since the Euclidean norm of $\mathbb{R}^d$ is continuous, by the Lemma 5.4.3, the function

$$\eta^0 : \Omega \longrightarrow \mathbb{R} \quad \eta^0(\omega) := \max_{x\in F(\omega)} |x|,$$

is $\mathcal{F}$-measurable and thus its class η is an element of L^0_+. This shows that $S \subset B_\varepsilon(0)$, where $\varepsilon := \eta \wedge 1 \in L^0_{++}$. Thus $\sup_{\eta\in S} |\eta| < \infty$.

$3 \Rightarrow 1$: Construct $(\xi_n) \subset S$ as in the proof of $3 \Rightarrow 1$ of Theorem 5.4.1. Since $\sup_{\eta\in S} |\eta| < \infty$, for each $n \in \mathbb{N}$, we can choose a representatives ξ^0_n of ξ_n so that $\{\xi^0_n(\omega) \colon n \in \mathbb{N}\}$ is bounded in $\mathbb{R}^d$ for each $\omega \in \Omega$. Then

$$F : \Omega \rightrightarrows \mathbb{R}^d \quad \omega \mapsto \overline{\{\xi^0_m(\omega) \colon m \in \mathbb{N}\}}.$$

meets the requirements. □

Notice that, in view of the theorem above, the correspondence $F \mapsto X_F$ defines a bijection between stably compact subsets of $L^0(\mathbb{R}^d)$ and measurable compact-valued mapping $F : \Omega \rightrightarrows \mathbb{R}^d$ with $\mathrm{dom}(F) = \Omega$.

A *normal integrand* is a function $\lambda : \Omega \times \mathbb{R}^d \to \overline{\mathbb{R}}$ such that its epigraph mapping

$$\Omega \rightrightarrows \mathbb{R}^d \times \mathbb{R} \quad \omega \mapsto \{(x, r) \in \mathbb{R}^d \times \mathbb{R} \colon \lambda(\omega, x) \leq r\}$$

is measurable and closed-valued. We say that a normal integrand λ is *ω-wise proper* if the functional $x \mapsto \lambda(\omega, x)$ is proper for all ω. Notice, that if λ is ω-wise proper, then the effective domain of the epigraph mapping is Ω.

Lemma 5.4.4. *[99, Proposition 14.28] If $\lambda : \Omega\times\mathbb{R}^d \to \overline{\mathbb{R}}$ is a normal integrand and $\eta \in L^0(\mathbb{R}^d)$, then the map $\omega \mapsto \lambda(\omega, \eta^0(\omega))$ is measurable whenever η^0 is a representative of η.*

The lemma above allows us to introduce the following:

Given a normal integrand $\lambda : \Omega \times \mathbb{R}^d \to \overline{\mathbb{R}}$, we define the map $f_\lambda : L^0(\mathbb{R}^d) \to \bar{L}^0((-\infty, \infty])$, where f_λ sends any η to the class of $\lambda(\cdot, \eta^0(\cdot))$ in $\bar{L}^0$.

Theorem 5.4.3. *If $\lambda : \Omega \times \mathbb{R}^d \to \overline{\mathbb{R}}$ is an ω-wise proper normal integrand, then $f_\lambda : L^0(\mathbb{R}^d) \to \bar{L}^0((-\infty, \infty])$ is a proper sequentially lower semi-continuous with the local property.*

Moreover, if $f : L^0(\mathbb{R}^d) \to \bar{L}^0((-\infty, \infty])$ is a proper sequentially lower semi-continuous and has the local property, there exists an ω-wise proper normal integrand λ such that $f = f_\lambda$. If λ' is another normal integrand satisfying that $f = f_{\lambda'}$, then, for any $x \in \mathbb{R}^d$, $\lambda(\omega, x) = \lambda'(\omega, x)$ for a.e. ω.

Proof. Let $\lambda : \Omega \times \mathbb{R}^d \to \overline{\mathbb{R}}$ be a normal integrand. Inspection shows that f_λ has the local property.

Since λ is ω-wise proper, we have that the epigraph mapping has effective domain equal to Ω. The epigraph mapping is also measurable closed-valued mapping. Since the epigraph mapping is measurable and closed-valued, due to Lemma 5.4.1, we can find a selection, and then we can conclude that f_λ is proper.

Further, if $V_\xi(f_\lambda) = \{\eta \in L^0(\mathbb{R}^d) \colon f_\lambda(\eta) \le \xi\}$ is non-empty, then $V_\xi(f_\lambda)$ is the set of selections of the closed-valued mapping

$$\Omega \rightrightarrows \mathbb{R}^d \quad \omega \mapsto \{x \in \mathbb{R}^d \colon \lambda(\omega, x) \le \xi^0(\omega)\},$$

where ξ is represented by ξ^0. By Theorem 5.4.1, we conclude that f_λ is sequentially lower semi-continuous.

Now, let $f : L^0(\mathbb{R}^d) \to \bar{L}^0((-\infty, \infty])$ be proper sequentially lower semi-continuous and with the local property. Consider its epigraph

$$\mathrm{epi}(f) = \{(\eta, \xi) \in L^0(\mathbb{R}^d) \times L^0 \colon f(\eta) \le \xi\}.$$

Since f is proper, $\mathrm{epi}(f)$ is non-empty; since f has the local property, $\mathrm{epi}(f)$ is stable; since f is sequentially lower semi-continuous, $\mathrm{epi}(f)$ is closed. As in Theorem 5.4.1, we can find a sequence $((\eta_n, \xi_n))_n$ in $\mathrm{epi}(f)$ such that the measurable closed-valued mapping

$$F_0 : \Omega \rightrightarrows \mathbb{R}^d \times \mathbb{R} \quad \omega \mapsto \overline{\{(\eta_n^0(\omega), \xi_n^0(\omega)) \colon n \in \mathbb{N}\}}$$

satisfies $X_{F_0} = \mathrm{epi}(f)$.

Enumerate $\mathbb{Q}^+ = \{q_1, q_2, \ldots\}$. Notice that $((\xi_n, \eta_n + q_k))_{n,k\in\mathbb{N}}$ is contained in $\mathrm{epi}(f)$. Thus the measurable closed-valued mapping

$$F : \Omega \rightrightarrows \mathbb{R}^d \times \mathbb{R} \quad \omega \mapsto \overline{\{(\eta_n^0(\omega), \xi_n^0(\omega) + q_k) \colon n, k \in \mathbb{N}\}}$$

satisfies that $X_F = \mathrm{epi}(f)$. Besides, if $(x, s) \in F(\omega)$, then $\{x\} \times [s, \infty) \subset F(\omega)$. For any $\omega \in \Omega$ and $x \in \mathbb{R}^d$, we define $F_x(\omega) := \{r \in \mathbb{R} \colon (x, r) \in F(\omega)\}$, which is of the form $F_x(\omega) = [r_x(\omega), \infty)$ for some $r_x(\omega) \in \mathbb{R}$. Consequently, one has that $F(\omega) = \{(x, s) \in \mathbb{R}^d \times \mathbb{R} \colon r_x(\omega) \le s\}$.

Then $\lambda : \Omega \times \mathbb{R}^d \to \overline{\mathbb{R}}$, with

$$\lambda(\omega, x) := \begin{cases} r_x(\omega), & \text{if } F_x(\omega) \neq \emptyset; \\ \infty, & \text{otherwise.} \end{cases}$$

is an ω-wise proper normal integrand whose epigraph mapping is precisely F.

Let us prove that $f = f_\lambda$. Indeed, take $\eta \in L^0(\mathbb{R}^d)$. Define $a := \{f(\eta) < \infty\}$ and fix some representative $A \in \mathcal{F}$ of a. Take representatives η^0 and $f(\eta)^0$ of η and $f(\eta)$, respectively. Since $X_F = \mathrm{epi}(f)$, we have that $(\eta^0(\omega), f(\eta)^0(\omega)) \in F(\omega)$ for a.e. $\omega \in A$. Therefore, $f(\eta)^0(\omega) \geq r_{\eta^0(\omega)}(\omega) = \lambda(\omega, \eta^0(\omega))$ for a.e. ω.

On the other hand, we have that

$$(\eta^0(\omega), \lambda(\omega, \eta^0(\omega))) = (\eta^0(\omega), r_{\eta^0(\omega)}(\omega)) \in F(\omega) \quad \text{for a.e. } \omega \in A.$$

The map $\omega \mapsto \lambda(\omega, \eta^0(\omega))$ is measurable (Lemma 5.4.4). Take its class $\nu \in \bar{L}^0$. Then $1_a(\eta, \nu) \in 1_a X_F = 1_a \mathrm{epi}(f)$. This means that $f(\eta) \leq \nu$ on a. Consequently, taking representatives $f(\eta)^0(\omega) \leq \lambda(\omega, \xi^0(\omega))$ for a.e. ω.

Finally, if λ' also satisfies that $f_{\lambda'} = f$, then, for $x \in \mathbb{R}^d$, we consider its class in $L^0(\mathbb{R}^d)$, which is denoted again by x. Then, since $f_{\lambda'}(x) = f(x)$, one has that $\lambda'(\omega, x) = \lambda(\omega, x)$ for a.e. ω. □

The conclusion is that the assignment $\lambda \mapsto f_\lambda$ defines a bijection between ω-wise normal integrands and proper sequentially lower semi-continuous functions from $L^0(\mathbb{R}^d)$ to $\bar{L}^0((-\infty, \infty])$ with the local property. This exhibits that proper sequentially lower semi-continuous functions are a suitable substitute for normal integrands when we wish to study stochastic control by using the tools of L^0-convex analysis.

5.5 A new proof of the Dalang-Morton-Willinger theorem

The so-called Dalang-Morton-Willinger theorem is the version of the celebrated First Fundamental Theorem of Asset Pricing for finite markets and discrete Finite time (see [26]). This result claims that there is no arbitrage if, and only if, the price process is a martingale with respect to an equivalent probability measure. This statement was first time proved by Harrison and Pliska under assumption of finite probability space [59]. The Dalang-Morton-Willinger theorem is a deep generalization of the Harrison-Pliska theorem, and has been widely accepted as one of the most important results in the arbitrage pricing theory. In literature it can be found different proofs of this result. Among others, it must be highlighted the following: Schachermayer [102], Rogers [100], Kabanov and Kramkov [69], Jacod and Shiryaev [62], and Kabanov and Stricker [70].

In the following, we will provide a new proof of this result which is based on the module approach. In the same spirit than the Bellman principle used previously to optimize stochastic control, we will reduce a multi-period problem to a one-period problem so that the module machinery can be applied. Then we apply the module analogue of the Kreps-Yan theorem 4.1.1 and a module analogue of orthogonal decompositions on finite-dimensional Euclidean spaces.

Let $S := (S_t)_{t=0}^T = (S_t^1, \ldots, S_t^d)_{t=0}^T$ be an $\mathbb{R}^d$-valued stochastic process which is adapted to our filtration $(\mathcal{F}_t)_{t=0}^T$. This process models the normalized (with respect to some numéraire) price of all risky assets of a financial market. We also consider trading strategies on the assets of the market. Precisely speaking, a trading strategy is a $\mathbb{R}^d$-valued stochastic process $\vartheta := (\vartheta_t)_{t=1}^T = (\vartheta_t^1, \ldots, \vartheta_t^d)_{t=1}^T$ which is predictable (i.e. ϑ_t is $\mathcal{F}_{t-1}$-measurable for $t = 1, \ldots, T$). Let Θ denote the set of trading strategies.

For given $\vartheta \in \Theta$, the gain process is given by the following discrete stochastic integral:

$$\vartheta \cdot S := \sum_{t=1}^T \langle \vartheta_t, \Delta S_t \rangle \quad \text{with } \Delta S_t := S_t - S_{t-1},$$

which is $\mathcal{F}_T$-measurable.

The set of *super-replicable contingent claims* is given by

$$C_T := \left\{\xi \in L^0_T \colon \xi = \vartheta \cdot S \text{ with } \vartheta \in \Theta\right\} - L^0_{T,+},$$

which is a convex cone.

The no-arbitrage condition is defined as follows:

$$C_T \cap L^0_{T,+} = \{0\}.$$

The Dalang-Morton-Willinger theorem asserts that the condition above is equivalent to the existence of an equivalent probability measure for which the process S is a martingale.

Let us consider the following sets

$$A_t := \left\{x \in L^0_{t+1} \colon x = \langle \vartheta, \Delta S_{t+1}\rangle,\ \vartheta \in L^0_t(\mathbb{R}^d)\right\} - L^0_{T,+}.$$

That is, A_t is the set of all elements of L^0_T which are dominated by some $\langle \vartheta, \Delta S_{t+1}\rangle$ with $\vartheta \in L_t(\mathbb{R}^d)$. Note that A_t is an L^0_t-convex L^0_t-cone.

We have the following statement of the Dalang-Morton-Willinger theorem:

Theorem 5.5.1. *The following conditions are equivalent:*

1. $C_T \cap L^0_{T,+} = \{0\}$*;*
2. $A_t \cap L^0_{T,+} = \{0\}$ *for all* $t = 0, \dots, T-1$*;*
3. $A_t \cap L^0_{T,+} = \{0\}$ *and* A_t *is sequentially closed for all* $t = 0, \dots, T-1$*;*
4. *there exist probability measures* $Q_0, \dots, Q_T$ *with* $Q_0 = P$ *such that*

$$Q_t \sim P,\ Q_t|_{\mathcal{F}_{t-1}} = Q_{t-1}|_{\mathcal{F}_{t-1}}, \text{ and } E_{Q_t}[S_t|\mathcal{F}_{t-1}] = S_{t-1} \quad \text{for all } t = 1, \dots, T;$$

5. *there exists a probability measure* Q*,* $Q \sim P$ *such that* S *is martingale.*

Let us see the proof of this statement. But first, we need to make some observations.

For each $t = 0, \dots, T-1$, let

$$E_t := \left\{\vartheta \in L^0_t(\mathbb{R}^d) \colon \langle \vartheta, \Delta S_{t+1}\rangle = 0\right\},$$

which is a $\mathcal{F}_t$-stable L^0_t-subspace of the euclidean L^0_t-module $L^0_t(\mathbb{R}^d_t)$. We can consider its orthogonal complement:

$$E_t^\perp := \left\{x \in L^0_t(\mathbb{R}^d) \colon \langle x, y\rangle = 0 \text{ for all } y \in E_t\right\}.$$

As a consequence of Proposition 5.1.6, we have the following decomposition $L^0_t(\mathbb{R}^d) = E_t \oplus E_t^\perp$, i.e. $L^0_t(\mathbb{R}^d) = E_t + E_t^\perp$ and $E_t \cap E_t^\perp = \{0\}$; and a orthogonal projection:

$$\pi_t : L^0_t(\mathbb{R}^d) \to E_t^\perp.$$

We use the following notation: if $1 \le p \le \infty$, we set

$$L^p_{t,T} := L^p_{\mathcal{F}_t}(\mathcal{F}_T), \quad L^p_{t,T,+} := L^p_{t,T} \cap L^0_+(\mathcal{F}_T), \quad L^p_{t,T,++} := L^p_{t,T} \cap L^0_{++}(\mathcal{F}_T).$$

Lemma 5.5.1. *If* $S \subset L^0_T$ *is sequentially closed, then* $S \cap L^1_{t,T}$ *is* L^0_t*-norm closed.*

Proof. Set $S_1 := S \cap L^1_{t,T}$. If $x \in \overline{S_1}^{\|\cdot|\mathcal{F}_t\|_1}$, then for all $n \in \mathbb{N}$, we can find $x_n \in S_1$ such that

$$E_P[|x - x_n| \wedge 1|\mathcal{F}_t] \leq E_P[|x - x_n||\mathcal{F}_t] \leq \frac{1}{n}.$$

In particular,

$$E_P[|x - x_n| \wedge 1] = E_P[E_P[|x - x_n| \wedge 1|\mathcal{F}_t]] \leq \frac{1}{n} \quad \text{ for all } n \in \mathbb{N}.$$

Thin implies that (x_n) converges in probability to x and thus we can find a subsequence (x_{n_k}) such that $\lim_k x_{n_k} = x$ a.s.. Since S is sequentially closed, we obtain that $x \in S_1$. $\square$

Let us turn to the proof of the main result.

Proof of Theorem 5.5.1:

$1 \Rightarrow 2$ is clear.

$5 \Rightarrow 1$: Let Q be an equivalent martingale measure. Then, if $\vartheta \cdot S = \sum_{t=1}^T \langle \vartheta_t, \Delta S_t \rangle \geq 0$, we have

$$E_Q[|\vartheta \cdot S|] = E_Q[\vartheta \cdot S] = \sum_{t=1}^T E_Q[\langle \vartheta_t, \Delta S_t \rangle] = \sum_{t=1}^T E_Q[E_Q[\langle \vartheta_t, \Delta S_t \rangle | \mathcal{F}_{t-1}]] = 0.$$

It follows that $\vartheta \cdot S = 0$.

$3 \Rightarrow 4$: Let us define a starting probability $Q_0 := P$. Now, suppose that Q_t has been defined for $0 \leq t \leq T-1$, and let us construct Q_{t+1} we proceed as follows:

Let $\eta := \max\limits_{i=1,\dots,d} |\Delta S^i_{t+1}|$ and $\nu := E_{Q_t}[e^{-\eta}|\mathcal{F}_t]$. Define

$$P'(A) := E_{Q_t}\left[1_a \frac{e^{-\eta}}{\nu}\right] \quad \text{ for } A \in \mathcal{F}_T.$$

This defines a probability measure (in fact, $dP' = \frac{e^{-\eta}}{\nu} dQ_t$). Further, if $A \in \mathcal{F}_t$, then

$$P'(A) = E_{Q_t}\left[1_a \frac{e^{-\eta}}{\nu}\right] = E_{Q_t}\left[1_a \frac{1}{\nu} E_{Q_t}[e^{-\eta}|\mathcal{F}_t]\right] = E_{Q_t}[1_a] = Q_t(A).$$

This means that $P'|_{\mathcal{F}_t} = Q_t|_{\mathcal{F}_t}$.

Let us replace the underlying probability P by P'. Note that our hypothesis 3 does not change if one replaces the probability P by other equivalent probability P'. By doing so, one has $\Delta S_{t+1} \in L^1_{t,T}$. Indeed,

$$E_{P'}\left[|\Delta S_{t+1}||\mathcal{F}_t\right] = E_{Q_t}\left[|\Delta S_{t+1}| \frac{dP'}{dQ_t} |\mathcal{F}_t\right] = \frac{E_{Q_t}\left[|\Delta S_{t+1}| e^{-\eta} |\mathcal{F}_t\right]}{\nu} \leq \frac{e^{-1}}{\nu} < \infty.$$

The $\mathcal{F}_t$-stable L^0_t-convex L^0_t-cone $A^1_t := A_t \cap L^1_{t,T}$ is L^0_t-norm closed due to Lemma 5.5.1. Since $A^1_t \cap L^1_{t,T,+} = \{0\}$ and $-L^1_{t,T,+} \subset A^1_t$, by Theorem 4.1.1, there exists $y \in L^\infty_{t,T,++}$ with $E_{P'}[y|\mathcal{F}_t] = 1$, such that

$$E_{P'}[yz|\mathcal{F}_t] \leq 0 \quad \text{ for all } z \in A^1_t.$$

Let Q_{t+1} be with $dQ_{t+1} = y dP'$. Then Q_{t+1} is equivalent to P, $Q_{t+1}|_{\mathcal{F}_t} = P'|_{\mathcal{F}_t} = Q_t|_{\mathcal{F}_t}$ and $E_{Q_{t+1}}[z|\mathcal{F}_t] \leq 0$ for all $z \in A^1_t$. Put $\vartheta := \operatorname{sgn}(E_{Q_{t+1}}[\Delta S_{t+1}|\mathcal{F}_t])$. We obtain that

$$0 \geq E_{Q_{t+1}}[\vartheta \cdot \Delta S_{t+1}|\mathcal{F}_t] = \vartheta \cdot E_{Q_{t+1}}[\Delta S_{t+1}|\mathcal{F}_t] = \sum_{i=1}^d |E_{Q_{t+1}}[\Delta S^i_{t+1}|\mathcal{F}_t]| \geq 0.$$

This implies that $E_{Q_{t+1}}[\Delta S_{t+1}|\mathcal{F}_t] = 0$, and the result follows.

$4 \Rightarrow 5$: It is not difficult to show that, if we find such a finite sequence of probabilities, then Q_T is an equivalent martingale probability measure.

$2 \Rightarrow 3$: Fixed $t \in \{0, \ldots, T-1\}$, let us prove that A_t is sequentially closed.

Take a sequence $(x_n) \subset A_t$ with $\lim_n x_n = x$ a.s.. For each $n \in \mathbb{N}$, put $x_n = \langle \vartheta_n, \Delta S_{t+1}\rangle - \xi_n$ with $\vartheta_n \in L^0_t(\mathbb{R}^d)$ and $\xi_n \in L^0_{T,+}$. Notice that we can suppose $\vartheta_n \in E_t^\perp$ by replacing ϑ_n by $\pi_t(\vartheta_n)$.

We claim that $\liminf_n |\vartheta_n| < \infty$. Indeed, suppose that $a := \{\liminf |\vartheta_n| = \infty\} \in \bar{\mathcal{F}}_t \setminus \{0\}$. We can argue separately on some representative A of a as an autonomous probability space. Thus, we can assume $A = \Omega$. Due to Proposition 5.1.2, we can take a sequence $\mathfrak{n}_1 < \mathfrak{n}_2 < \ldots$ in $L^0_t(\mathbb{N})$ with $\lim_m |\vartheta_{\mathfrak{n}_m}| = \infty$ and $|\vartheta_{\mathfrak{n}_m}| > 0$ for all $m \in \mathbb{N}$.

Now, the sequence $(\frac{\vartheta_{\mathfrak{n}_m}}{|\vartheta_{\mathfrak{n}_m}|})_{m\in\mathbb{N}}$ is L^0_t-norm bounded. Thus, by Proposition 5.1.3, we can find a sequence $\mathfrak{m}_1 < \mathfrak{m}_2 < \ldots$ in $L^0(\mathbb{N})$ such that

$$\lim_k \frac{\vartheta_{\mathfrak{n}_{\mathfrak{m}_k}}}{|\vartheta_{\mathfrak{n}_{\mathfrak{m}_k}}|} = \eta \text{ a.s. in } L^0_t(\mathbb{R}^d) \quad \text{and} \quad \lim_k |\vartheta_{\mathfrak{n}_{\mathfrak{m}_k}}| = \infty \text{ a.s..}$$

Further, being $E_t^\perp$ sequentially closed and $\mathcal{F}_t$-stable, we have that $\eta \in E_t^\perp$. Thus:

$$0 \leq \lim_k \frac{\xi_{\mathfrak{n}_{\mathfrak{m}_k}}}{|\vartheta_{\mathfrak{n}_{\mathfrak{m}_k}}|} = \lim_k \langle \frac{\vartheta_{\mathfrak{n}_{\mathfrak{m}_k}}}{|\vartheta_{\mathfrak{n}_{\mathfrak{m}_k}}|}, \Delta S_{t+1}\rangle - \lim_k \frac{x_{\mathfrak{n}_{\mathfrak{m}_k}}}{|\vartheta_{\mathfrak{n}_{\mathfrak{m}_k}}|} = \langle \eta, \Delta S_{t+1}\rangle - 0 \quad \text{a.s..}$$

Since $A_t \cap L^0_{T,+} = \{0\}$, necessarily $\langle \eta, \Delta S_{t+1}\rangle = 0$. This means that $\eta \in E_t$, and thus $\eta \in E_t \cap E_t^\perp = \{0\}$, hence $\eta = 0$. But this is a contradiction, given that $|\eta| = 1$.

We conclude that $\liminf_n |\vartheta_n| \in L^0_t$. Finally, again by Proposition 5.1.2, we can take a sequence $\mathfrak{n}_1 < \mathfrak{n}_2 < \ldots$ in $L^0(\mathbb{N})$ so that $\lim_k \vartheta_{\mathfrak{n}_k} = \vartheta$ a.s. in $L^0_t(\mathbb{R}^d)$. Then, we obtain $x = \langle \vartheta, \Delta S_{t+1}\rangle - \xi$, for some $\xi \in L^0_{T,+}$, and the result follows.

$\square$

Comments 5.5.1. *It was kindly communicated by M. Kupper that a proof of the Dalang-Morton-Willinger theorem based on L^0-convex analysis was independently obtained by Arno Blaas in his master thesis entitled 'Conditional analysis applied on the Fundamental Theorem of Asset Pricing', whose details the author does not know (see [17]).*

In addition, we thank W. Schachemayer for pointing out that an earlier module approach to partial aspects of the Fundamental Theorem of Asset Pricing was provided in [106].

Appendices

Appendix A

Basic set theory

A.1 The First-order Language of Set Theory

In this section we will describe the usual language of set theory. We have a precisely determined collection of symbols $\mathcal{L}$. Among these symbols, we have:

1. A countable collection of *free variables* which specify sets;
2. A countable collection of *bounded variables* which run over sets;
3. two predicates: the equality symbol $=$ and the membership symbol $\in$;
4. logical connectives: $\wedge$ (and), $\vee$ (or) and a negation symbol $\neg$;
5. quantifier symbols: $\forall$ (for all), $\exists$ (there exists);
6. brackets: (,).

This describes the lexicon for the formal language of set theory, which is adequate for expressing any mathematical statement.

In the following, we will use $u, v, w, u_1, u_2, \ldots$ as metavariables whose domain is the collection of free variables. Similarly, we will use $x, y, z, x_1, x_2, x_3, \ldots$ as metavariables whose domain is the collection of bounded variables.

The syntax of the language consists of *formulas* which are constructed as follows:

An expression of the form $(u = v)$ or $(u \in v)$ where u, v are free variables is called an *atomic formula.*

In general, the collection of formulas is defined as follows:

1. Any atomic formula is a formula;
2. if φ is a formula, then $\neg\varphi$ is a formula;
3. if φ, ψ are formulas, then $(\varphi \wedge \psi)$ and $(\varphi \vee \psi)$ are formulas;
4. if φ is a formula, then $(\forall x)\varphi(x)$ and $(\exists x)\varphi(x)$ are formulas, where $\varphi(x)$ is obtained by replacing each occurrence in φ of some free variable u in φ by the bounded variable x.

A formula which does not contain free variables is called a *sentence*.

In the writing of the formulas we will omit some parenthesis. By convention, we will assume that the predicate symbols $\in$ and $=$ have preference over the logic connectives, so we will not write parenthesis in the atomic components of formulas.

Although this syntax allows the formulation of any mathematical sentence, the language $\mathcal{L}$ is usually enriched by introducing some abbreviations for constructible formulas which make $\mathcal{L}$ to resemble more and more the usual mathematical discussion. For instance, $(\varphi \Rightarrow \psi)$ abbreviates $(\neg\varphi \vee \psi)$. Also, $(\varphi \Leftrightarrow \psi)$ abbreviates $((\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi))$.

Next, we give some examples of abbreviated texts in the language of set theory, which gives a more intuitive form to the language closer to the usual mathematical discourse:

$(\exists! x)\varphi(x)$	abbreviates $(\exists x)\varphi(x) \wedge (\forall x)(\forall y)((\varphi(x) \wedge \varphi(y)) \Rightarrow x = y)$;
$(\forall x \in u)\varphi(x)$	abbreviates $(\forall x)(x \in u \Rightarrow \varphi(x))$;
$(\exists x \in u)\varphi(x)$	abbreviates $(\exists x)(x \in u \wedge \varphi(x))$;
$u \subset v$	abbreviates $(\forall x)(x \in u \Rightarrow y \in v)$;
$u = \mathcal{P}(v)$	abbreviates $(\forall x)(x \in u \Rightarrow x \subset v)$;
$u = \emptyset$	abbreviates $(\forall x)(x \in u \Leftrightarrow x \neq x)$;
$u = \bigcup v$	abbreviates $(\forall x)(x \in u \Leftrightarrow (\exists y)(x \in y \wedge y \in v))$;
$u = \{v\}$	abbreviates $(\forall x)(x \in u \Leftrightarrow x = v)$;
$u = \{v, w\}$	abbreviates $(\forall x)(x \in u \Leftrightarrow (x = v \vee x = w))$;
$u = (v, w)$	abbreviates $(\forall x)(x \in u \Leftrightarrow (x = \{v\} \vee x = \{u, v\}))$;
$u = v \times w$	abbreviates $(\forall x \in u)(\exists y \in v)(\exists z \in w)(x = (y, z))$;
$u \in \mathrm{fun}(v, w)$	abbreviates $(u \subset v \times w) \wedge (\forall x \in v)(\exists! y)((x, y) \in u)$.

A.2 The Axioms of Zermelo-Fraenkel with the axiom of choice

The Zermelo-Fraenkel set theory with the axiom of choice (ZFC) is the collection of sentences in the first-order language $\mathcal{L}$ —which we call theorems— that are derived by means of the common *logical axioms* and *rules of inference* of predicate calculus (see, eg [67, Chapter 2]) from the (non-logical) axioms listed below:

1. Extensionality: If x and y have the same elements, then $x = y$:

$$(\forall x)(\forall y)(x = y \Leftrightarrow (\forall z)(z \in x \Leftrightarrow z \in y)).$$

2. Pairing: For any x and y there exists a set $\{x, y\}$ that contains exactly x and y:

$$(\forall x)(\forall y)(\exists z)(z = \{x, y\}).$$

3. Union: For any x there exists a set $y = \bigcup x$, the union of all the elements of x:

$$(\forall x)(\exists y)\left(y = \bigcup x\right).$$

4. Power set: For any x there exists a set $y = \mathcal{P}(x)$ which is the set of all subsets of x:
$$(\forall x)(\exists y)(y = \mathcal{P}(x)).$$
5. Regularity: Every non-empty set has an $\in$-minimal element:
$$(\forall x)(x \neq \emptyset \Rightarrow (\exists y \in x)(x \cap y = \emptyset)).$$
6. Infinity: There is an infinite set:
$$(\exists x)(x \neq \emptyset \wedge (\forall y \in x)(y \cup \{y\} \in x)).$$
7. Comprehension Schema: If $\varphi(u)$ is a formula (with one free variable u), then for any x there exists the set that contains all those $z \in x$ that satisfies $\varphi(z)$:
$$(\forall x)(\exists y)(z \in y \Leftrightarrow z \in x \wedge \varphi(z)).$$
8. Replacement Schema: Ranges of functions exist; if $\varphi(u, v)$ is a formula (with two free variables u, v), then
$$(\forall x)((\forall y \in x)((\exists! z)\varphi(y, z)) \Rightarrow (\exists t)(\forall z)(z \in t \Leftrightarrow (\exists y \in x)\varphi(y, z))).$$
9. Choice: Choice function exists
$$(\forall x \neq \emptyset)(\emptyset \notin x \Rightarrow (\exists f \in \mathrm{fun}(x, \bigcup x))((\forall y \in x)(f(y) \in y))).$$

Although we work in ZFC, which 'talks' about sets, we introduce the informal notion of class. If $\varphi(u, v_1, \ldots, v_n)$ is a formula with free variables $u, v_1, \ldots, v_n$ $(n \geq 0)$, the symbol
$$C = \{u \colon \varphi(u, v_1, \ldots, v_n)\}$$
is called a class of sets, or simply a class and the record $u \in C$ abbreviates the formula $\varphi(u, v_1, \ldots, v_n)$. In this case, we say that the class C is *definable* from $v_1, \ldots, v_n$; if $\varphi(x)$ has no parameters v_i, then the class C is definable.

We say that two classes C, D are equal whenever $x \in C$ if and only if $x \in D$. The *universal class*, or *universe*, is the class of sets
$$V = \{x \colon x = x\}.$$
Every set can be seen as a class; namely, if S is a set, consider the formula $x \in S$ and the class
$$\{x \colon x \in S\}.$$
Recall that a set is uniquely determined by its elements due to the Axiom of Extensionality.

ZFC theory is the formalization of the language we use to talk about sets. In the following, we will work with ZFC, but we are not concerned with the details of formalization as we assume that the reader knows that usual set-theoretic reasoning can be formalized in ZFC. Therefore we will talk about sets and classes as in the usual mathematical discourse.

A.3 Ordered sets

A partial order is a relation $\leq$ on a set X is the following properties are verified for all $x, y, z \in X$:

1. $x \leq x$;
2. $x \leq y$ and $y \leq x$ implies $x = y$;
3. $x \leq y$ and $y \leq z$ implies $x \leq z$.

The pair $(X, \leq)$ is called a partially ordered set.

An *upper bound* (resp. lower bound) of a subset Y of a set X is an element $x \in X$ such that $y \leq x$ (resp. $x \leq x$) for all $y \in Y$. A *supremum* (resp. infimum) of Y is an upper (resp. lower) bound $z \in X$ of Y such that $z \leq x$ (resp. $x \leq z$) for every upper (resp. lower) bound $x \in X$ of Y.

A partially ordered set $(X, \leq)$ is said to be totally ordered if for every $x, y \in X$

Recall that a *well-ordered set* is a totally ordered set $(X, \leq)$ with the property that any non-empty subset has a smallest element, that is, there exists $x_0 \in X$ such that $x_0 \leq x$ for all $x \in X$.

We have the following result:

Theorem A.3.1. *(Well-ordering theorem) Every set can be well-ordered.*

A.4 Ordinal numbers, transfinite induction and transfinite recursion

The concept of ordinal number is essential to study infinite sets. In addition, ordinal numbers enable the method of *transfinite induction*.

Definition A.4.1. *A set α is an* ordinal number *if the following is satisfied:*

1. *$\beta \in \alpha$ implies $\beta \subset \alpha$;*
2. *α is totally ordered by the relation $x \leq y$ whenever either $x = y$ or $x \in y$.*

Notice that due to the regularity axiom $x = y$ and $x \in y$ cannot happen at the same time.
If α and β are ordinals, we write $\alpha < \beta$ whenever $\alpha \in \beta$.
If α is an ordinal, then $\alpha \cup \{\alpha\}$ is also an ordinal.

The following properties can be checked from the definition of ordinal:

- $\emptyset$ is an ordinal;
- If α is an ordinal and $\beta \in \alpha$, then β is an ordinal;
- If α, β are ordinals with $\alpha \neq \beta$, then either $\alpha < \beta$ or $\beta < \alpha$;
- For each ordinal α, $\alpha = \{\beta \colon \beta < \alpha\}$;
- For every α, $\alpha \cup \{\alpha\}$ is an ordinal and $\alpha \cup \{\alpha\} = \inf\{\beta \colon \beta > \alpha\}$;
- If C is a non-empty class of ordinals, then $\bigcap C$ is an ordinal and $\bigcap C = \inf C$;
- If X is a non-empty set of ordinals, then $\bigcup X$ is an ordinal and $\bigcup X = \sup X$.

We define $\alpha + 1 := \alpha \cup \{\alpha\}$ the *successor* of α.

If α is not the successor of any ordinal, then $\alpha = \sup\{\beta \colon \beta < \alpha\} = \bigcup \alpha$ is an ordinal which is called a *limit ordinal.* The ordinal $\emptyset$ is a limit ordinal $\emptyset = \bigcup \emptyset$.

Proposition A.4.1. *Every ordinal is either successor or limit.*

We can define the intersection of all inductive non-empty set, which is also an ordinal and is called the set of *natural numbers* $\mathbb{N}$. The ordinals less than $\mathbb{N}$ are called *finite ordinals* or natural numbers. Specifically, one sets $0 := \emptyset$, $1 := 0 + 1$, $2 := 1 + 1$, etc.

Theorem A.4.1. *Every well-ordered set is isomorphic to a unique ordinal number.*

Theorem A.4.2. *(Transfinite induction) Let C be a class of ordinals and assume that*

1. *$\emptyset \in C$;*
2. *if $\alpha \in C$, then $\alpha + 1 \in C$;*
3. *$\emptyset \neq \alpha$ is a limit ordinal and $\beta \in C$ for all $\beta < \alpha$, then $\alpha \in C$.*

Then C is the class of all ordinals.

Theorem A.4.3. *(Transfinite recursion) Let G be a function on the class V then there is a unique class function F on* Ord *such that*

$$F(\alpha) = G(F|_\alpha) \quad \textit{for all } \alpha \in \mathrm{Ord},$$

where $F|_\alpha$ is the restriction of F to α.

We define the canonical well-ordering of the class $\mathrm{Ord} \times \mathrm{Ord}$ of ordinal pairs:

$$\begin{aligned}(\alpha, \beta) < (\gamma, \delta) \text{ whenever } & \text{ either } \max\{\alpha, \beta\} < \max\{\gamma, \delta\}, \\ & \text{ or } \max\{\alpha, \beta\} = \max\{\gamma, \delta\} \text{ and } \alpha < \gamma, \\ & \text{ or } \max\{\alpha, \beta\} = \max\{\gamma, \delta\} \text{ and } \alpha = \gamma \text{ and } \beta < \delta.\end{aligned}$$

A.5 The cumulative hierarchy of sets

Suppose that we have a universe of sets V which fulfils the axioms of ZFC, that is, V a domain of the sets of our model. We will see next that the elements of V can be classified into a transfinite hierarchy.

The universe V satisfies the axiom of regularity, that is,

$$\forall x(x \neq \emptyset \to (\exists y \in x) x \cap y = \emptyset).$$

This axiom makes impossible the existence of infinite sequences

$$x_0 \ni x_1 \ni x_2 \ni x_3 \ni \dots$$

Indeed, otherwise one can consider $x := \{x_0, x_1, x_2, \dots\}$ and apply the axiom, reaching a contradiction.

Note that this also implies that there is no set with $x \in x$ and there are no "cycles" $x_0 \in x_1 \in \dots \in x_n \in x_0$.

Definition A.5.1. *A set T is said to be* transitive *if $x \in T$ implies $x \subset T$.*

Lemma A.5.1. *For every set x there exists a transitive set T with $x \subset T$.*

Proof. By induction, we define $T_0 := x$, $T_{n+1} := \bigcup T_n$. It suffices to take $T := \bigcup_{n \in \mathbb{N}} T_n$. $\square$

The transitive closure of x is defined by

$$T(x) := \bigcap \{T \colon x \subset T,\ T \text{ is transitive}\}.$$

Notice that $T(x)$ is the smallest transitive set that contains x.

Lemma A.5.2. *Every non-empty class $\mathcal{C}$ has an $\in$-minimal element.*

Proof. Let x be an arbitrary element of $\mathcal{C}$. If x and $\mathcal{C}$ have no element in common, the x is an $\in$-minimal element. Suppose that x and $\mathcal{C}$ have common elements, consider $T := T(x)$ and define

$$z := \{u \in T \colon u \text{ is in } \mathcal{C}\}.$$

One has that $z \neq \emptyset$, and due to the axiom of regularity, there exists $u \in z$ such that $u \cap z = \emptyset$. It follows that u and $\mathcal{C}$ have no element in common. Otherwise, if $v \in u$ and v is an element of $\mathcal{C}$, then $v \in T$ as T is transitive. Since v is also in $\mathcal{C}$, it follows that $v \in z$. This yields that $v \in u \cap z = \emptyset$, a contradiction. We conclude that u is an $\in$-minimal element of $\mathcal{C}$. $\square$

Let us construct the cumulative hierarchy of sets. By transfinite induction, we construct $V_0 := \emptyset$, $V_{\alpha+1} := \mathcal{P}(V_\alpha)$, and $V_\alpha := \bigcup_{\beta<\alpha} V_\beta$ if α is a limit ordinal.

By transfinite induction it can be verified that the sets V_α have the following properties:

(i) Each V_α is transitive;

(ii) if $\alpha < \beta$, then $V_\alpha \subset V_\beta$;

(iii) $\alpha \subset V_\alpha$.

We have the following:

Proposition A.5.1. *For each x in V there exists an ordinal α such that $x \in V_\alpha$.*

Proof. Define that class $\mathcal{C}$ of all x that are not in any V_α. By contradiction suppose that $\mathcal{C}$ is not empty. Then, by Lemma A.5.2, we can find a $\in$-minimal element x. This means that x is in $\mathcal{C}$ and for any $z \in x$ there is an ordinal α with $z \in V_\alpha$. By the axiom of replacement, there exists an ordinal γ such that $x \subset \bigcup_{\alpha<\gamma} V_\alpha$. Hence $x \subset V_\gamma$ and so $x \in V_{\gamma+1}$, a contradiction. $\square$

A.6 Boolean algebras

Definition A.6.1. *A* Boolean algebra *is a set $\mathcal{A}$ with at least two elements* 0 *and I, endowed with two binary operations $\vee$ and $\wedge$ and an unitary operation $\cdot^c$. The Boolean operations satisfy the following axioms for all $a, b, c \in \mathcal{A}$:*

1. *(commutativity) $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$;*
2. *(associativity) $a \vee (b \vee c) = (a \vee b) \vee c$ and $a \wedge (b \wedge c) = (a \wedge b) \wedge c$;*
3. *(distributivity) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ and $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$;*
4. *(absorption) $a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$;*

5. *(complementation)* $a \vee a^c = I$ *and* $a \wedge a^c = 0$.

An *algebra of sets* is a collection of subsets of a given non-empty set that is closed under finite unions, finite intersections and complements. We have that any algebra of sets is a Boolean algebra.

The following properties can be derived from the axioms above:

$$a \vee a = a, \quad a \wedge a = a, \quad a \vee 0 = a, \quad a \wedge 0 = 0, \quad a \vee I = I, \quad a \wedge I = a,$$

and the De Morgan laws:

$$(a \vee b)^c = a^c \wedge b^c, \quad (a \wedge b)^c = a^c \vee b^c.$$

We can also define a partial ordering $\leq$ on $\mathcal{A}$ by putting

$$a \leq b \quad \text{whenever } a \wedge b = a.$$

In addition, I is the greatest element of $\mathcal{A}$ and 0 is the least element.

An *ideal* of $\mathcal{A}$ is a subset $\mathcal{I}$ of $\mathcal{A}$ which satisfies:

1. $0 \in \mathcal{I}$, $I \notin \mathcal{I}$;
2. $a, b \in \mathcal{I}$ implies $a \vee b \in \mathcal{I}$;
3. $a \in \mathcal{A}$, $b \in \mathcal{I}$ and $a \leq b$ imply $a \in \mathcal{I}$.

The set $\{0\}$ is called *trivial ideal*. Given $a \in \mathcal{A} \setminus \{I\}$, we define the *ideal generated by* a to be $\mathcal{A}_a := \{b \in \mathcal{A} \colon b \leq a\}$.

Given a non-empty subset X of $\mathcal{A}$, the supremum (resp. infimum) of Y is denoted by $\bigvee Y$ (resp. $\bigwedge Y$), if exists. We say that $\mathcal{A}$ is σ-complete if every non-empty countable subset of $\mathcal{A}$ has a supremum and an infimum. We say that $\mathcal{A}$ is complete if every non-empty subset of $\mathcal{A}$ has a supremum and an infimum.

Let $\mathcal{A}, \mathcal{B}$ be Boolean algebras. A mapping $\lambda : \mathcal{A} \to \mathcal{B}$ is called a *homomorphism* if:

1. $\lambda(0) = 0$, $\lambda(I) = I$,
2. $\lambda(a \vee b) = \lambda(a) \vee \lambda(b)$, $\lambda(a \wedge b) = \lambda(a) \wedge \lambda(b)$, and $\lambda(a^c) = \lambda(a)^c$.

A bijective homomorphism is called an *isomorphism*.

Notice that if λ is a homomorphism, then $\lambda(a) \leq \lambda(b)$ whenever $a \leq b$.

Theorem A.6.1. *(Stone's Representation theorem) Every Boolean algebra is isomorphic to an algebra of sets.*

A non-empty subset $\mathcal{Y}$ of a Boolean Algebra $\mathcal{A}$ is said to be *disjoint* if for any $a, b \in \mathcal{Y}$ with $a \neq b$ it is satisfied that $a \wedge b = 0$.

A Boolean algebra $\mathcal{A}$ has the *countable chain condition*, or is ccc, if any disjoint subset $\mathcal{Y}$ of $\mathcal{A}$ is at most countable.

Definition A.6.2. *A* measure algebra *is a pair* $(\mathcal{A}, \mu)$ *where* $\mathcal{A}$ *is a* σ*-complete Boolean algebra and* $\mu : \mathcal{A} \to [0, \infty]$ *is a function which satisfies:*

1. *$\mu(a) = 0$ if and only if $a = 0$;*

2. *$\mu\left(\bigvee a_n\right) = \sum_{n\geq 1} \mu(a_n)$ for every sequence (a_n) of elements of $\mathcal{A}$ such that $a_n \wedge a_m = 0$ whenever $n \neq m$.*

We will say that a measure algebra $(\mathcal{A}, \mu)$ is a *probability algebra* when $\mu(I) = 1$.

Given a measure space (X, Σ, μ) we consider the equivalence relation on Σ given by

$$a \sim b \quad \text{whenever } \mu(a \setminus b) = 0 = \mu(b \setminus a).$$

Then the quotient set $\bar{\Sigma}$ is a σ-complete Boolean algebra. If $\pi : \Sigma \to \bar{\Sigma}$ denotes the projection, then we have a function $\bar{\mu} : \bar{\Sigma} \to [0, \infty]$ defined by $\bar{\mu}(\pi(a)) := \mu(a)$ for all $a \in \Sigma$, and $(\bar{\Sigma}, \bar{\mu})$ is a measure algebra such that for any non-empty sequence (a_n) in $\mathcal{A}$ the following holds:

$$\pi\left(\bigcup_n a_n\right) = \bigvee_n \pi(a_n), \quad \pi\left(\bigcap_n a_n\right) = \bigwedge_n \pi(a_n).$$

If $(\Omega, \mathcal{F}, P)$ is a probability space, then its associated measure algebra $(\bar{\mathcal{F}}, \bar{P})$ is a probability algebra.

Proposition A.6.1. *Let (Ω, Σ, μ) be a σ-finite measure space. Then the associated measure algebra $(\bar{\Sigma}, \bar{\mu})$ is ccc.*

Let $(\mathcal{A}, \mu)$, $(\mathcal{B}, \nu)$ be measure algebras. An isomorphism $\lambda : \mathcal{A} \to \mathcal{B}$ of Boolean algebras such that $\mu(a) = \nu(\lambda(a))$ for all $a \in \mathcal{A}$ is called an *isomorphism* of measure algebras.

Theorem A.6.2. *(The Stone representation of a measure algebra) For any measure algebra $(\mathcal{A}, \mu)$ there exists a measure space (X, Σ, μ) such that $(\mathcal{A}, \mu)$ and $(\bar{\Sigma}, \bar{\mu})$ are isomorphic measure algebras.*

Corollary A.6.1. *For any probability algebra $(\mathcal{A}, \mu)$ there exists a probability space $(\Omega, \mathcal{F}, P)$ such that $(\mathcal{A}, \mu)$ and $(\bar{\mathcal{F}}, \bar{P})$ are isomorphic probability algebras.*

Appendix B

Some conventional proofs for conditional versions of fundamental results of functional analysis

In this chapter, we use the techniques of conditional set theory to provide conventional proofs for some of the conditional versions of classical theorems established in Section 3.3 of Chapter 3.

Namely, we prove conditional versions of the Baire category theorem, the Uniform boundedness principle, the Eberlein-Šmulian theorem, the Amir-Lindenstrauss theorem, the James' compactness theorem, and a perturbed variation of the latter. For this purpose, we provide a comprehensive study of conditional weak topologies with a special emphasis on results evolving conditional compactness.

We also use the conditional perturbed James' compactness theorem to prove the version of the Jouini-Schachermayer-Touzi theorem for conditional risk measures established in Chapter 4.

B.1 Preliminaries

In the following, we will fixed some notation. Given a conditional set $\mathbf{X}$ and a non-empty subset Y of X, we define the *stable hull* of Y to be

$$\mathfrak{s}(Y) := \left\{ \sum x_i|a_i \colon (a_i) \in p(I),\ (x_i) \subset Y \right\}.$$

Clearly, $\mathfrak{s}(Y)$ is a stable subset of X, therefore we can define the corresponding conditional subset, which is denoted by $\boldsymbol{\mathfrak{s}}(Y)$.

From time to time, some conditional subsets will be required to be defined by describing their conditional elements. For instance, suppose that $\varphi(u)$ is certain formula with a free variable u which we suppose to run over the conditional elements of $\mathbf{X}$. Also, assume that $\varphi(\mathbf{x})$ holds for a conditional element $\mathbf{x} \in \mathbf{X}$. Since the family $\{\mathbf{x} \in \mathbf{X} \colon \varphi(\mathbf{x})\}$ is not a conditional set (it does not contain elements of the for $x|a$ with $a < I$), we employ the following formal set-builder notation for conditional subsets:

$$[\mathbf{x} \in \mathbf{X} \colon \varphi(\mathbf{x})] = \boldsymbol{\mathfrak{s}}(\{x \colon \varphi(\mathbf{x})\}).$$

We will focus our study on conditionally normed spaces. Given a conditionally normed space $(\mathbf{E}, \|\cdot\|)$, for any $\mathbf{x}$ we define its support to be

$$\mathrm{supp}(\mathbf{x}) := \bigvee\{a \in \mathcal{A}\colon \ \|\,\mathbf{x}\,\|\,|a > \mathbf{0}|a\}.$$

The support of $\mathbf{E}$ is defined by

$$\mathrm{supp}(\mathbf{E}) := \bigvee_{\mathbf{x}\in\mathbf{E}} \mathrm{supp}(\mathbf{x}).$$

Notice that, by stability, the support of $\mathbf{E}$ is attained, that is, $\mathrm{supp}(\mathbf{E}) = \mathrm{supp}(\mathbf{x})$ for some $\mathbf{x} \in \mathbf{E}$.

In the following, by a conditionally normed space we will mean a conditionally normed space with support I, unless otherwise specified.

If $(\mathbf{E}, \|\cdot\|)$ is a conditionally normed space, we define

$$\mathbf{B_E} := [\mathbf{x} \in \mathbf{E}\colon \ \|\,\mathbf{x}\,\| \leq \mathbf{1}] \quad \text{the conditional unit ball of } \mathbf{E},$$

and

$$\mathbf{S_E} := [\mathbf{x} \in \mathbf{E}\colon \ \|\,\mathbf{x}\,\| = \mathbf{1}] \quad \text{the conditional unit sphere of } \mathbf{E},$$

which is well-defined as the $\mathrm{supp}(\mathbf{E}) := I$.

Suppose that $(\mathbf{r_k})_{\mathbf{1}\,\leq\,\mathbf{k}\,\leq\,\mathbf{n}}$ and $(\mathbf{x_k})_{\mathbf{1}\,\leq\,\mathbf{k}\,\leq\,\mathbf{n}}$ are conditionally finite families in $\mathbf{R}$ and $\mathbf{E}$, respectively, where $\mathbf{n}$ is a conditional natural number with $n = \sum n_i|a_i$, $(a_i) \in p(I)$ and $(n_i) \subset \mathbb{N}$. We define

$$\sum_{\mathbf{1}\,\leq\,\mathbf{k}\,\leq\,\mathbf{n}} \mathbf{r_k x_k}$$

to be the conditional element of $\mathbf{E}$ generated by $\sum_i \left(\sum_{k=1}^{n_i} r_k x_k\right)|a_i$.

If $(\mathbf{r_n})$ is a conditional sequence in $\mathbf{R}$, we define

$$\sum_{\mathbf{n}\,\geq\,\mathbf{1}} \mathbf{r_n} := \mathbf{lim_n} \sum_{\mathbf{1}\,\leq\,\mathbf{k}\,\leq\,\mathbf{n}} \mathbf{r_k},$$

provided that the above conditional limit exists.

Given a conditional real number $\mathbf{r} \in \mathbf{R}$, we will denote by $\mathbf{r^{-1}}$ the conditional real number generated by

$$r^{-1}|\mathrm{supp}(\mathbf{r}) + 0|\mathrm{supp}(\mathbf{r}).$$

Throughout we will use conditional versions of classical theorems which are proven by conventional means in [32]. For all these auxiliary result we will give an exact reference.

B.2 Conditional versions of Baire Category theorem and Uniform Boundedness Principle

In this section we constructively prove the conditional analogues of the Baire Category theorem and Uniform Boundedness Principle established in Chapter 3.

Proof of Theorem 3.3.4:

Define

$$d := \bigvee \left\{ a \in \mathcal{A}\colon \text{ there exists } \mathbf{B_r}(\mathbf{x})|a \sqsubset \mathbf{E_n}|a \text{ for some } \mathbf{n} \in \mathbf{N} \text{ and } \mathbf{r} \in \mathbf{R}^{++} \right\}.$$

We claim that d is attained. Indeed, take $(a_i) \in p(d)$ with $\mathbf{B}_{\mathbf{r}_i}(\mathbf{x}_i)|a_i \sqsubset \mathbf{E}_{\mathbf{n}_i}|a_i$ for each i. Fix $x_0 \in E$. If we take $r = \sum r_i|a_i + 1|d^c$, $x = \sum x_i|a_i + x_0|d^c$ and $n = \sum n_k|a_i + 1|d^c$, it follows that $\mathbf{B_r}(\mathbf{x})|d \sqsubset \mathbf{E_n}|d$.

If $d = I$ we are done as d is attained. In other case, we may assume w.l.o.g. $d = 0$ by consistently arguing on d^c.

Let $\mathbf{E}_1 := \mathbf{E_1}$. We claim that $\mathbf{E}_1^{\sqsubset}$ is on I. Indeed, if $\mathbf{E}_1^{\sqsubset}$ is on d_1, then since $\mathbf{E} = \mathbf{E}_1 \sqcup \mathbf{E}_1^{\sqsubset}$, using the distributivity of $\sqcap$ over $\sqcup$, we have that

$$\mathbf{E}|d_1^c = \mathbf{E} \sqcap \mathbf{E}|d_1^c = (\mathbf{E}_1 \sqcap \mathbf{E}|d_1^c) \sqcup (\mathbf{E}_1^{\sqsubset} \sqcap \mathbf{E}|d_1^c) = (\mathbf{E}_1|d_1^c) \sqcup \mathbf{E}|0 = \mathbf{E}_1|d_1^c.$$

Then, $\mathbf{E}|d_1^c = \mathbf{E}_1|d_1^c$, in particular $\mathbf{B}_1(x)|d_1^c \sqsubset \mathbf{E}_1|d_1^c$ for some fixed $\mathbf{x} \in \mathbf{E}$, which implies that $d_1^c = 0$, because $d = 0$. Given that $\mathbf{E}_1^{\sqsubset}$ is conditionally open and on I, it must therefore contain a ball $\mathbf{B}_1 := \mathbf{B}_{\mathbf{r}_1}(\mathbf{x}_1)$ with $\mathbf{0} < \mathbf{r}_1 < \mathbf{1/2}$.

Now look at $\mathbf{E}_2 := \mathbf{E_2}$. We claim that $\mathbf{E}_2^{\sqsubset} \sqcap \mathbf{B}_{\mathbf{r}_1/\mathbf{2}}(\mathbf{x}_1)$ is on I. Indeed, suppose that $\mathbf{E}_2^{\sqsubset} \sqcap \mathbf{B}_{\mathbf{r}_1/\mathbf{2}}(\mathbf{x}_1)$ is on d_2. Then one has that $\mathbf{B}_{\mathbf{r}_1/\mathbf{2}}(\mathbf{x}_1)|d_2^c \sqsubset \mathbf{E}_2|d_2^c$, and necessarily $d_2^c = 0$, since $d = 0$. Being $\mathbf{E}_2^{\sqsubset}$ conditionally open, it follows that $\mathbf{E}_2^{\sqsubset} \sqcap \mathbf{B}_{\mathbf{r}_1/\mathbf{2}}(\mathbf{x}_1)$ conditionally contains a conditional ball $\mathbf{B}_2 := \mathbf{B}_{\mathbf{r}_2}(\mathbf{x}_2)$ with $\mathbf{0} < \mathbf{r}_2 < \mathbf{1/4}$.

By induction, we obtain a sequence $\mathbf{B}_k := \mathbf{B}_{\mathbf{r}_k}(\mathbf{x}_k)$ of conditional balls such that $\mathbf{0} < \mathbf{r}_k < \mathbf{1/2^k}$, $\mathbf{B}_{k+1} \sqsubset \mathbf{B}_{\mathbf{r}_k/\mathbf{2}}(\mathbf{x}_k)$, and $\mathbf{B}_k \sqcap \mathbf{E}_k = \mathbf{E}|0$ for all $k \in \mathbb{N}$.

For each $\mathbf{n} = n|I \in \mathbf{N}$ with $n = \sum_{i \in I} n_i|a_i$, where $\{a_i\} \in p(1)$ and $n_i \in \mathbb{N}$ for all i, we define $\mathbf{x_n} := x_n|I$ and $\mathbf{z_n} := z_n|I$ with $x_n := \sum x_{n_i}|a_i$ and $r_n := \sum r_{n_i}|a_i$. Then $(\mathbf{x_n})_{\mathbf{n}\in\mathbf{N}}$ and $(\mathbf{r_n})_{\mathbf{n}\in\mathbf{N}}$ are conditional sequences.

For $\mathbf{n}, \mathbf{m} \in \mathbf{N}$ with $n = \sum_i n_i|b_i$ and $m = \sum_j m_j|c_j$, let us define $d_{i,j} := b_i \wedge c_j$ and $a := \vee\{b \in \mathcal{A}\colon m|b = n|b\}$. Then

$$d(x_n, x_m) = d(x_{n\vee m}, x_{n\wedge m}) \leq 0|a + \sum_{i,j} \left(\sum_{k=n_i\wedge m_j}^{(n_i\vee m_j)-1} d(x_k, x_{k+1}) \right) |a^c \wedge d_{i,j} \leq \frac{1}{2^{m\wedge n}}.$$

Consequently, $(\mathbf{x_n})_{\mathbf{n}\in\mathbf{N}}$ is a conditional Cauchy sequence, and given that $\mathbf{E}$ is conditionally complete, it conditionally converges to some $\mathbf{x} \in \mathbf{E}$.

Now, since for all $\mathbf{n} < \mathbf{m}$

$$\mathbf{d}(\mathbf{x_n}, \mathbf{x}) \leq \mathbf{d}(\mathbf{x_n}, \mathbf{x_m}) + \mathbf{d}(\mathbf{x_m}, \mathbf{x}) < \frac{\mathbf{r_n}}{\mathbf{2}} + \mathbf{d}(\mathbf{x_m}, \mathbf{x})$$

and $\mathbf{lim_m}\, \mathbf{d}(\mathbf{x_m}, \mathbf{x}) = \mathbf{0}$, it follows that $\mathbf{d}(\mathbf{x_n}, \mathbf{x}) \leq \frac{\mathbf{r_n}}{\mathbf{2}}$. In particular, it means that $\mathbf{x} \in \mathbf{B}_{\mathbf{r_n}}(\mathbf{x_n})$ for every $\mathbf{n} \in \mathbf{N}$.

On the other hand, since $\mathbf{E} = \bigsqcup_{\mathbf{n}\in\mathbf{N}} \mathbf{E_n}$, we have that $\mathbf{x} \in \mathbf{E_n}$ for some $\mathbf{n}$. Let us take $b \in \mathcal{A}$, $b \neq 0$, such that $n|b = k|b$ for some $k \in \mathbb{N}$. Then $x|b \in \mathbf{E}_k \sqcap \mathbf{B}_k$. But this is a contradiction.

□

Proof of Theorem 3.3.5:

For each $\mathbf{n} \in \mathbf{N}$ we define the following conditional set

$$\mathbf{E_n} := \left[\mathbf{x} \in \mathbf{E}\colon \sup_{\mathbf{T}\in\mathbf{S}} \| \mathbf{T}(\mathbf{x}) \| \leq \mathbf{n} \right].$$

We can consider the conditionally countable family $(\mathbf{E_n})_{\mathbf{n}\in\mathbf{N}}$.

Then it is clear that $\mathbf{E} = \bigsqcup_{\mathbf{n}\in\mathbf{N}} \mathbf{E_n}$.

Let us show that $\mathbf{E_n}$ is conditionally closed. For that, let us take a conditional sequence $(\mathbf{x_k})_{\mathbf{k}\in\mathbf{N}}$ in $\mathbf{E_n}$ which conditionally converges to $\mathbf{x}$, and let us show that $\mathbf{x} \in \mathbf{E_n}$. Indeed, arguing by way of contradiction, assume that there exists $c \in \mathcal{A}$, $c \neq 0$, and $\mathbf{T} \in \mathbf{S}$ so that $\| \mathbf{T}(\mathbf{x}) \| |c > \mathbf{n}|c$. Let us suppose $c = I$, by consistently arguing on c. Since $(\| \mathbf{T}(\mathbf{x_k}) \|)_{\mathbf{k}}$ conditionally converges to $\| \mathbf{T}(\mathbf{x}) \|$, there exists some $\mathbf{k}' \in \mathbf{N}$ with $\| \mathbf{T}(\mathbf{x_{k'}}) \| \geq \mathbf{n}$, which is a contradiction.

By Theorem 3.3.4, it follows that there exists a conditional ball $\mathbf{B_r}(\mathbf{z})$ such that $\mathbf{B_r}(\mathbf{z}) \sqsubset \mathbf{E_n}$ for some $\mathbf{n} \in \mathbf{N}$.

Let $\mathbf{x} \in \mathbf{B_E}$ and $\mathbf{T} \in \mathbf{S}$. It follows

$$\| \mathbf{T}(\mathbf{x}) \| = \frac{\mathbf{1}}{\mathbf{r}} \| \mathbf{T}(\mathbf{z} - \mathbf{r}\mathbf{x}) - \mathbf{T}(\mathbf{z}) \| \leq \frac{\mathbf{1}}{\mathbf{r}} (\| \mathbf{T}(\mathbf{z} - \mathbf{r}\mathbf{x}) \| + \| \mathbf{T}(\mathbf{z}) \|) \leq \frac{\mathbf{1}}{\mathbf{r}} \mathbf{2n}.$$

□

B.3 Some basic results on conditional weak topologies

An important result for the conditional weak topologies is the conditional version of Alaoglu theorem:

Theorem B.3.1. *[32, Theorem 5.10] Let $\boldsymbol{E}$ be a conditionally normed space, then conditional dual unit ball $\boldsymbol{B}_{\boldsymbol{E}^*}$ is conditionally compact with respect to $\boldsymbol{\sigma}(\boldsymbol{E}^*, \boldsymbol{E})$.*

Now let us turn to see some results related to conditional weak topologies.

Proposition B.3.1. *Let $(\boldsymbol{E}, \boldsymbol{\mathcal{T}})$ be a conditional locally convex space, and let $\boldsymbol{A} \sqsubset \boldsymbol{E}$ be conditionally convex. Then $\overline{\boldsymbol{A}}^{\boldsymbol{\mathcal{T}}} = \overline{\boldsymbol{A}}^{\boldsymbol{\sigma}(\boldsymbol{E},\boldsymbol{E}^*)}$.*

Proof. Clearly, $\overline{\mathbf{A}}^{\boldsymbol{\mathcal{T}}} \sqsubset \overline{\mathbf{A}}^{\boldsymbol{\sigma}(\mathbf{E},\mathbf{E}^*)}$.

For the reverse conditional inclusion, let us take $\mathbf{x} \in \overline{\mathbf{A}}^{\boldsymbol{\sigma}(\mathbf{E},\mathbf{E}^*)}$.

Set

$$b := \bigvee \left\{ a \in \mathcal{A} \colon x|a \in \overline{\mathbf{A}}^{\boldsymbol{\mathcal{T}}} \right\}.$$

Since b is attained, if $b = I$ we get the result. If not, we can consistently argue on b^c, and suppose w.l.o.g. $b = 0$.

If so, we have that $[\mathbf{x}] \sqcap \overline{\mathbf{A}}^{\boldsymbol{\mathcal{T}}} = \mathbf{E}|0$. Besides, $[\mathbf{x}]$ is conditionally compact, and $\overline{\mathbf{A}}^{\boldsymbol{\mathcal{T}}}$ is conditionally convex and conditionally closed with respect to $\boldsymbol{\mathcal{T}}$.

Then, the conditional extension theorem (see [32, Theorem 5.3]) yields $\mathbf{x}^* \in \mathbf{E}^*$ and $\mathbf{r} \in \mathbf{R}^{++}$ such that

$$\mathbf{x}^*(\mathbf{x}) + \mathbf{r} \leq \mathbf{x}^*(\mathbf{y}), \quad \text{for all } \mathbf{y} \in \overline{\mathbf{A}}^{\boldsymbol{\mathcal{T}}}. \tag{B.1}$$

Let us define $\mathbf{B} := [\mathbf{y} \in \mathbf{E} \colon \mathbf{x}^*(\mathbf{y}) \geq \mathbf{x}^*(\mathbf{x}) + \mathbf{r}]$, which is a conditionally $\boldsymbol{\sigma}(\mathbf{E}, \mathbf{E}^*)$-closed subset.

But (B.1) means that $\mathbf{A} \sqsubset \mathbf{B}$ and $\mathbf{x} \in \mathbf{B}^{\sqsubset}$. This is a contradiction as $\mathbf{x}$ is in the conditional weak closure of $\mathbf{A}$. □

The following result gives the *natural conditional embedding* of a conditionally normed space $\mathbf{E}$ into the conditional second dual $\mathbf{E}^{**} := (\mathbf{E}^*)^*$.

Theorem B.3.2. *Let $\boldsymbol{E}$ be a conditionally normed space. For each $\boldsymbol{x} \in \boldsymbol{E}$, let us define the conditional function $\boldsymbol{T_x} : \boldsymbol{E}^* \to \boldsymbol{R}$ with $\boldsymbol{T_x}(\boldsymbol{x}^*) := \boldsymbol{x}^*(\boldsymbol{x})$. Then $\boldsymbol{T_x} \in \boldsymbol{E}^{**}$ for each $\boldsymbol{x} \in \boldsymbol{E}$, and the conditional function $\boldsymbol{j} : \boldsymbol{E} \to \boldsymbol{E}^{**}$ given by $\boldsymbol{j}(\boldsymbol{x}) := \boldsymbol{T_x}$ is a conditional isometry, i.e. $\|\boldsymbol{x}\| = \|\boldsymbol{j}(\boldsymbol{x})\|$ for all $\boldsymbol{x} \in \boldsymbol{E}$.*

Proof. It suffices to show that for all $\mathbf{x} \in \mathbf{E}$ it holds that $\|\mathbf{x}\| = \mathbf{sup}\,[|\mathbf{x}^*(\mathbf{x})| : \mathbf{x}^* \in \mathbf{B}_{\mathbf{E}^*}]$.

First, let us show that there exists $\mathbf{x}^* \in \mathbf{B}_{\mathbf{E}^*}$ such that $\mathbf{x}^*(\mathbf{x}) = \|\mathbf{x}\|$. Indeed, we have that the conditional function $\mathbf{x}_0^* : \mathbf{span}_{\mathbf{R}}[\mathbf{x}] \to \mathbf{R}$ with $\mathbf{x}_0^*(\mathbf{rx}) := \mathbf{r}\,\|\mathbf{x}\|$ is well defined and $\mathbf{x}_0^* \in (\mathbf{span}_{\mathbf{R}}[\mathbf{x}])^*$. Furthermore, notice that $\mathbf{x}_0^*(\mathbf{y}) \leq \|\mathbf{y}\|$ for all $\mathbf{y} \in \mathbf{span}_{\mathbf{R}}[\mathbf{x}]$. By the conditional version of the Hahn-Banach extension theorem [32, Theorem 5.3], there exists $\mathbf{x}^* \in \mathbf{E}^*$ which extends $\mathbf{x}_0^*$ and such that $|\,\mathbf{x}^*(\mathbf{y})\,| \leq \|\mathbf{y}\|$ for all $\mathbf{y} \in \mathbf{E}$. We conclude that $\mathbf{x}^* \in \mathbf{B}_{\mathbf{E}^*}$. Moreover, notice that, in fact, $\|\mathbf{x}^*\| = \mathbf{1}$.

Finally, let $\mathbf{r} := \mathbf{sup}\,[|\,\mathbf{z}^*(\mathbf{x})\,| : \mathbf{z}^* \in \mathbf{B}_{\mathbf{E}^*}]$. Then it is clear that $\mathbf{r} \leq \|\mathbf{x}\| = \mathbf{x}^*(\mathbf{x}) \leq |\,\mathbf{x}^*(\mathbf{x})\,| \leq \mathbf{r}$, and we obtain what was asserted. ☐

Theorem B.3.3. *[Conditional version of Goldstine's theorem] Let $\boldsymbol{j} : \boldsymbol{E} \to \boldsymbol{E}^{**}$ be the natural conditional embedding. Then $\boldsymbol{j}(\boldsymbol{B_E})$ is conditionally $\boldsymbol{\sigma}(\boldsymbol{E}^{**}, \boldsymbol{E}^*)$-dense in $\boldsymbol{B}_{\boldsymbol{E}^{**}}$.*

Proof. Define $\mathbf{K} := \overline{\mathbf{j}(\mathbf{B_E})}^{\boldsymbol{\sigma}(\mathbf{E}^{**}, \mathbf{E}^*)}$. Since $\mathbf{B}_{\mathbf{E}}^{**}$ is conditionally $\boldsymbol{\sigma}(\mathbf{E}^{**}, \mathbf{E}^*)$-closed, it follows that $\mathbf{K} \sqsubset \mathbf{B}_{\mathbf{E}}^{**}$. Besides, $\mathbf{K}$ is conditionally convex; then, if we suppose that there exists $\mathbf{x}^{**} \in \mathbf{K}^{\sqsubset} \sqcap \mathbf{B}_{\mathbf{E}^{**}}$, the conditional hyperplane separation theorem (see [32, Theorem 5.5]) provides us with a conditionally $\boldsymbol{\sigma}(\mathbf{E}^{**}, \mathbf{E}^*)$-continuous linear function $\mathbf{f} : \mathbf{E}^{**} \to \mathbf{R}$ so that

$$\mathbf{sup}\,[\mathbf{f}(\mathbf{s}) : \mathbf{s} \in \mathbf{K}] < \mathbf{f}(\mathbf{x}^{**}).$$

In addition, we have $(\mathbf{E}^{**})^*[\boldsymbol{\sigma}(\mathbf{E}^{**}, \mathbf{E}^*)] = \mathbf{E}^*$ (see [73, Chapter 4, Corollary 4.48]). Consequently, we can find $\mathbf{x}^* \in \mathbf{E}^*$ with $\mathbf{f}(\mathbf{z}^{**}) = \mathbf{z}^{**}(\mathbf{x}^*)$ for all $\mathbf{z}^{**} \in \mathbf{E}^{**}$.

Then, being $\mathbf{B_E}$ conditionally balanced, it follows that

$$\mathbf{sup}\,[\mathbf{x}^*(\mathbf{x}) : \mathbf{x} \in \mathbf{B_E}] = \mathbf{sup}\,[|\,\mathbf{x}^*(\mathbf{x})\,| : \mathbf{x} \in \mathbf{B_E}] < \mathbf{f}(\mathbf{x}^{**}) = \mathbf{x}^{**}(\mathbf{x}^*). \tag{B.2}$$

But this is impossible given that

$$\mathbf{x}^{**}(\mathbf{x}^*) = |\,\mathbf{x}^{**}(\mathbf{x}^*)\,| \leq \|\mathbf{x}^{**}\|\,\|\mathbf{x}^*\| < \mathbf{x}^{**}(\mathbf{x}^*)\,\|\mathbf{x}^{**}\| \leq \mathbf{x}^{**}(\mathbf{x}^*).$$

Notice that the first equality is true because the conditional supremum (B.2) is conditionally greater than or equal to $\mathbf{0}$. ☐

B.4 Conditional versions of Eberlein-Šmulian and Amir-Lindenstrauss theorems

In this section we constructively prove the conditional analogues of the Eberlein-Šmulian and Amir-Lindenstrauss theorems established in Chapter 3.

Before proving these two theorems, we need to make a remark. We claim that $1 = \mathrm{supp}(\mathbf{E}^*) = \mathrm{supp}(\mathbf{E}^{**}) = \ldots$ Indeed, we assumed $\mathrm{supp}(\mathbf{E}) = I$. Then, we can choose $\mathbf{x} \in [\mathbf{0}]^{\sqsubset}$ with $\mathrm{supp}(\mathbf{x}) = I$. Reasoning as in the proof of Theorem B.3.2, we can find $\mathbf{x}^* \in \mathbf{E}^*$ such that $\mathbf{x}^*(\mathbf{x}) = \|\mathbf{x}\|$. In particular $\mathbf{x}^* \in [\mathbf{0}]^{\sqsubset}$. This proves that $\mathrm{supp}(\mathbf{E}^*) = I$. The same applies to $\mathbf{E}^{**} = (\mathbf{E}^*)^*$, and so on.

Now, let us turn to show some preliminary results:

Lemma B.4.1. *Let $\boldsymbol{E}$ be a conditionally normed space. Suppose that $\boldsymbol{K} \sqsubset \boldsymbol{E}$ is conditionally weakly compact, then $\boldsymbol{K}$ is conditionally bounded and conditionally norm closed.*

Proof. If $\mathbf{x}^* \in \mathbf{E}^*$, then $\mathbf{x}^*$ is a conditionally weakly continuous function, and $\mathbf{x}^*(\mathbf{K})$ is therefore conditionally weakly compact (see [32, Proposition 3.26]), hence it is a conditionally bounded subset of $\mathbf{R}$. Since $\mathbf{K}$ can be conditionally embedded into $\mathbf{E}^{**}$ (see Theorem B.3.2), we can apply Theorem 3.3.5 obtaining that $\mathbf{K}$ is conditionally bounded. □

Lemma B.4.2. *Let $\boldsymbol{E}$ be a conditionally normed space and suppose $\boldsymbol{K} \sqsubset \boldsymbol{E}$, conditionally bounded and conditionally weakly closed. Suppose that $\boldsymbol{j}: \boldsymbol{E} \to \boldsymbol{E}^{**}$ is the natural conditional embedding. Then, $\overline{\boldsymbol{j}(\boldsymbol{K})}^{\boldsymbol{\sigma}(\boldsymbol{E}^{**},\boldsymbol{E}^*)} \sqsubset \boldsymbol{j}(\boldsymbol{E})$ if, and only if, $\boldsymbol{K}$ is conditionally weakly compact.*

Proof. Suppose that $\overline{\mathbf{j}(\mathbf{K})}^{\sigma(\mathbf{E}^{**},\mathbf{E}^*)} \sqsubset \mathbf{j}(\mathbf{E})$. By the conditional theorem of Banach-Alaoglu (see Theorem B.3.1) we have that $\overline{\mathbf{j}(\mathbf{K})}^{\boldsymbol{\sigma}(\mathbf{E}^{**},\mathbf{E}^*)}$ is conditionally $\boldsymbol{\sigma}(\mathbf{E}^{**},\mathbf{E}^*)$-compact. Since $\boldsymbol{\sigma}(\mathbf{E}^{**},\mathbf{E}^*)|_{\mathbf{j}(\mathbf{E})} = \sigma(\mathbf{j}(\mathbf{E}),\mathbf{E}^*)$, it follows that

$$\mathbf{j}(\mathbf{K}) = \overline{\mathbf{j}(\mathbf{K})}^{\boldsymbol{\sigma}(\mathbf{j}(\mathbf{E}),\mathbf{E}^*)} = \overline{\mathbf{j}(\mathbf{K})}^{\boldsymbol{\sigma}(\mathbf{E}^{**},\mathbf{E}^*)} \sqcap \mathbf{j}(\mathbf{E}) = \overline{\mathbf{j}(\mathbf{K})}^{\sigma(\mathbf{E}^{**},\mathbf{E}^*)}.$$

We conclude that $\mathbf{K}$ is conditionally weakly compact. □

Definition B.4.1. *A conditional topological space $(\boldsymbol{X},\boldsymbol{\mathcal{T}})$ is said to be* conditionally separable *if there exists a conditionally countable subset $\boldsymbol{Y}$ of $\boldsymbol{X}$ such that sequence $(\boldsymbol{x_n})$ in $\boldsymbol{X}$ such that*

Definition B.4.2. *Let $(\boldsymbol{X}, \|\cdot\|)$ be a conditionally normed space. A conditional subset $\boldsymbol{Y}$ of $\boldsymbol{E}^*$ is* conditionally total *if $\boldsymbol{x} = \boldsymbol{0}$ whenever $\boldsymbol{x}^*(\boldsymbol{x}) = \boldsymbol{0}$ for all $\boldsymbol{x}^* \in \boldsymbol{Y}$.*

Lemma B.4.3. *Assume that $\boldsymbol{E}$ is conditionally Banach and conditionally separable. Then, there is a conditionally countable total subset $\boldsymbol{D}$ of $\boldsymbol{E}^*$.*

Proof. Put $\mathbf{C} := [\mathbf{x_n} \colon \mathbf{n} \in \mathbf{N}]$ for some sequence $(\mathbf{x_n})$ in $\mathbf{E}$. For each $n \in \mathbb{N}$, as argued in the proof of Theorem B.3.2, there exists some $\mathbf{x}^*_n \in \mathbf{B}_{\mathbf{E}^*}$ such that $\mathbf{x}^*_n(\mathbf{x}) = \|\mathbf{x}_n\|$. For any $\mathbf{n} \in \mathbf{N}$ with $n = \sum n_i|a_i$, where $(a_i) \in p(I)$ and $(n_i) \subset \mathbb{N}$, set $\mathbf{x}^*_{\mathbf{n}} := \sum \mathbf{x}^*_i|a_i$. We claim that $\mathbf{D} := [\mathbf{x}^*_{\mathbf{n}} \colon \mathbf{n} \in \mathbf{N}]$ is conditionally total. Let $\mathbf{x} \in \mathbf{X}$ with $\mathbf{x}^*_{\mathbf{n}} = \mathbf{0}$ for all $\mathbf{n} \in \mathbf{N}$. We have that $\mathbf{x} = \mathbf{0}$. Otherwise, suppose that $a := \operatorname{supp}(\mathbf{x}) > 0$. Consistently arguing on a, assume $a = I$. Then, since $\mathbf{C}$ is conditionally dense, for each $m \in \mathbb{N}$, we can find $\mathbf{n}_m \in \mathbf{N}$ so that $\|\mathbf{x}_{\mathbf{m}_n} - \mathbf{x}\| \leq \mathbf{1/m}$. For any $\mathbf{m} \in \mathbf{N}$ with $m = \sum m_i|a_i$, $(m_i) \subset \mathbb{N}$ and $(a_i) \in p(I)$, set $\mathbf{x}^*_{\mathbf{n_m}} := \sum \mathbf{x}^*_{\mathbf{n}_{m_i}}|a_i$. Then $\mathbf{lim_m}\, \mathbf{x}^*_{\mathbf{n_m}} = \mathbf{x}$. Therefore

$$\mathbf{lim_m}\, \mathbf{x}^*_{\mathbf{n_m}}(\mathbf{x}) = \mathbf{lim_m}(\mathbf{x}^*_{\mathbf{n_m}}(\mathbf{x} - \mathbf{x_{n_m}}) + \mathbf{x}^*(\mathbf{x_{n_m}})) = \mathbf{lim_m}\, \|\mathbf{x_{n_m}}\| = \|\mathbf{x}\| > \mathbf{0}.$$

Necessarily, there exists $\mathbf{m}_0$ such that $\|\mathbf{x}^*_{\mathbf{n_m}}\| > \mathbf{0}$, which is a contradiction. □

Lemma B.4.4. *If $\boldsymbol{E}$ is a conditionally normed space, and $(\boldsymbol{x_n})$ is a conditional sequence which conditionally weakly converges to $\boldsymbol{x} \in \boldsymbol{E}$, then $(\boldsymbol{x_n})$ is conditionally bounded. As a consequence, if $\boldsymbol{K}$ is conditionally weakly sequentially compact, then $\boldsymbol{K}$ is conditionally bounded.*

Proof. For the first part, we consider the conditional sequence $(\mathbf{x}^{**}_{\mathbf{n}})_{\mathbf{n}\in\mathbf{N}}$ with $\mathbf{x}^{**}_{\mathbf{n}} := \mathbf{j}(\mathbf{x_n})$, where $\mathbf{j}$ is the natural conditional embedding. This is a conditional sequence in $\mathbf{E}^{**}$. Since $(\mathbf{x_n})$ conditionally weakly converges to $\mathbf{x}$, for fixed $\mathbf{x}^* \in \mathbf{E}^*$, there exists $\mathbf{n}_0 \in \mathbf{N}$ so that $|\mathbf{x}^*(\mathbf{x_n})| \leq \mathbf{1} + |\mathbf{x}^*(\mathbf{x})|$ whenever $\mathbf{n} \geq \mathbf{n}_0$. Then, for each $\mathbf{n} \in \mathbf{N}$

$$|\mathbf{z_n}(\mathbf{x}^*)| = |\mathbf{x}^*(\mathbf{x_n})| \leq (\mathbf{1} + |\mathbf{x}^*(\mathbf{x})|) \vee \mathbf{max}[|\mathbf{x}^*(\mathbf{x_n})| \colon \mathbf{n} \leq \mathbf{n}_0].$$

Now, due to Theorem 3.3.5, we have that $(\mathbf{x}^{**}_{\mathbf{n}})$ is conditionally bounded. Consequently $(\mathbf{x_n})$ is conditionally bounded too, because $\mathbf{j}$ is a conditional isometry.

As for the second part, define

$$d := \bigvee \{a \in \mathcal{A} \colon \mathbf{K}|a \text{ is conditionally bounded } \}.$$

d is attained. So if $d = I$ we are done. In other case, we can assume that $d = 0$. If so, for each $n \in \mathbb{N}$ let us define

$$d_n := \bigvee \{a \in \mathcal{A} \colon \text{ exists } \mathbf{x} \in \mathbf{K} \text{ with } \|\mathbf{x}\| \,|a > \mathbf{n}|a\}.$$

Since $d = 0$, necessarily $d_n = I$ and it is attained, so we can pick $\mathbf{x}_n \in \mathbf{K}$, each $n \in \mathbb{N}$, such that $\|\mathbf{x}_n\| > \mathbf{n}$.

For $\mathbf{n} \in \mathbf{N}$ with $n = \sum n_i|a_i$, $(a_i) \in p(I)$ and $(n_i) \subset \mathbb{N}$, define $x_n := \sum x_{n_i}|a_i$. Then $(\mathbf{x_n})$ is a conditional sequence with $\|\mathbf{x_n}\| > \mathbf{n}$ for each $\mathbf{n} \in \mathbf{N}$. Due to the first part of the theorem, we find that the conditional sequence $(\mathbf{x_n})$ cannot have a conditionally weakly convergent subsequence. □

Lemma B.4.5. *Let $\boldsymbol{E}$ be a conditionally normed space. If $\boldsymbol{K} \sqsubset \boldsymbol{E}$ is conditionally weakly compact and there is a conditionally countable set $\boldsymbol{D} \sqsubset \boldsymbol{E}^*$ which is conditionally total, and such that $\|\boldsymbol{x}^*\| > \boldsymbol{0}$ for all $\boldsymbol{x}^* \in \boldsymbol{D}$, then $\boldsymbol{K}$ is conditionally metrizable.*

Proof. Put $\mathbf{D} = [\mathbf{x}^* \colon \mathbf{n} \in \mathbf{N}]$ for some conditional sequence $(\mathbf{x_n^*})$ in $\mathbf{E}^*$. We will construct a conditional metric on $\mathbf{K}$. Indeed, we define the conditional function $\mathbf{d} : \mathbf{E} \times \mathbf{E} \to \mathbf{R}$ given by

$$\mathbf{d}(\mathbf{x}, \mathbf{y}) := \sum_{\mathbf{n} \geq \mathbf{1}} \frac{\mathbf{1}}{\mathbf{2^n}} \frac{|\mathbf{x_n^*}(\mathbf{x} - \mathbf{y})|}{\|\mathbf{x_n^*}\|} \qquad \text{for } \mathbf{x}, \mathbf{y} \in \mathbf{E}.$$

By using that $\mathbf{K}$ is conditionally total, it can be verified that $\mathbf{d}$ is a conditional metric on $\mathbf{E}$. Besides, due to Lemma B.4.1, we have that $\mathbf{K}$ is conditionally bounded. Now, we consider the identity $\mathbf{Id} : \mathbf{K} \to \mathbf{K}$ as a conditional function between the conditional metric spaces $(\mathbf{K}, \sigma(\mathbf{E}, \mathbf{E}^*)|_{\mathbf{K}})$ and $(\mathbf{K}, \mathbf{d})$. Then $\mathbf{Id}$ is conditionally continuous.

Indeed, let $(\mathbf{x_j})_{\mathbf{j} \in \mathbf{J}}$ be a conditional net in $\mathbf{K}$ which conditionally converges to some $\mathbf{x} \in \mathbf{K}$ with respect to $\sigma(\mathbf{E}, \mathbf{E}^*)|_{\mathbf{K}}$.

For arbitrary $\mathbf{r} \in \mathbf{R}^{++}$, let $\mathbf{l} := \mathbf{sup}[\|\mathbf{x}\| \colon \mathbf{x} \in \mathbf{K}] \in \mathbf{R}$. Choose $\mathbf{m} \in \mathbf{N}$, $\mathbf{m} > \mathbf{1}$, with $\frac{\mathbf{1}}{\mathbf{2^m}} \leq \frac{\mathbf{r}}{\mathbf{2(1+l)}}$. Then

$$\sum_{\mathbf{n} \geq \mathbf{m}} \frac{\mathbf{1}}{\mathbf{2^n}} \frac{|\mathbf{x_n^*}(\mathbf{x_j} - \mathbf{x})|}{\|\mathbf{x_n^*}\|} \leq \frac{\mathbf{1}}{\mathbf{2^m}} \mathbf{l} \leq \mathbf{r}/\mathbf{2} \qquad \text{for all } \mathbf{j} \in \mathbf{J}.$$

Since $(\mathbf{x_j})$ conditionally converges to $\mathbf{x}$, we can also choose $\mathbf{j}_0 \in \mathbf{J}$ such that

$$\sum_{\mathbf{1} \leq \mathbf{n} \leq \mathbf{m-1}} \frac{\mathbf{1}}{\mathbf{2^n}} \frac{|\mathbf{x_n^*}(\mathbf{x_j} - \mathbf{x})|}{\|\mathbf{x_n^*}\|} \leq \mathbf{r}/\mathbf{2}, \qquad \text{for all } \mathbf{j} \in \mathbf{J} \text{ with } \mathbf{j} \geq \mathbf{j}_0.$$

It follows that $\mathbf{d}(\mathbf{x_i}, \mathbf{x}) \leq \mathbf{r}$ whenever $\mathbf{i} \geq \mathbf{i}_0$.

Finally, let us show that $\mathbf{Id}$ is a conditionally open function.

Let $\mathbf{O} \sqsubset \mathbf{K}$ be conditionally $\boldsymbol{\sigma}(\mathbf{E}, \mathbf{E}^*)|_{\mathbf{K}}$-open. Since $\mathbf{K}$ is conditionally $\boldsymbol{\sigma}(\mathbf{E}, \mathbf{E}^*)$-compact, it follows that $\mathbf{O}^{\sqsubset} \sqcap \mathbf{K}$ is conditionally $\boldsymbol{\sigma}(\mathbf{E}, \mathbf{E}^*)|_{\mathbf{K}}$-compact. Then, since $\mathbf{Id}$ is conditionally continuous $\mathbf{O}^{\sqsubset} \sqcap \mathbf{K}$ is conditionally compact in $(\mathbf{K}, \mathbf{d})$ (see [32, Proposition 3.26]), hence $\mathbf{O}$ is conditionally open in $(\mathbf{K}, \mathbf{d})$. □

Proof of Theorem 3.3.6:

Suppose that $\mathbf{K}$ is a conditionally weakly compact subset of $\mathbf{E}$, and let $(\mathbf{z_n})_{\mathbf{n}\in\mathbf{N}}$ be a conditional sequence in $\mathbf{K}$. Set $\mathbf{E}_0 := \overline{\mathbf{span_R}[\mathbf{z_n} \colon \mathbf{n}\in\mathbf{N}]}^{\boldsymbol{\sigma}(\mathbf{E},\mathbf{E}^*)}$. Then $\mathbf{E}_0$ is conditionally weakly closed in $\mathbf{E}$ and, consequently, $\mathbf{K}_0 := \mathbf{E}_0 \sqcap \mathbf{K}$ is conditionally $\boldsymbol{\sigma}(\mathbf{E}_0,\mathbf{E}_0^*)$-compact in the conditionally complete normed space $\mathbf{E}_0$. Such space is conditionally separable, because it conditionally contains the conditionally countable and dense subset $\mathbf{span_Q}[\mathbf{z_n} \colon \mathbf{n}\in\mathbf{N}]$. We can thus apply Lemma B.4.3; that is, $\mathbf{E}_0^*$ conditionally contains a conditionally countable total set.

From Lemma B.4.5 we know that $\mathbf{K}_0 \sqcap \mathbf{K}$ is conditionally metrizable with the conditional topology $\sigma(\mathbf{K}_0,\mathbf{E}^*)$. Since every conditionally compact metric space is conditionally sequentially compact (see [32, Theorem 4.6]), we have that $\mathbf{K}_0 \sqcap \mathbf{K}$ is conditionally sequentially $\boldsymbol{\sigma}(\mathbf{E}_0,\mathbf{E}_0^*)|_{\mathbf{K}_0}$-compact. In particular, if we choose a conditional subsequence $(\mathbf{z_{n_m}})_{\mathbf{m}\in\mathbf{N}}$ which conditionally converges in $\boldsymbol{\sigma}(\mathbf{E}_0,\mathbf{E}_0^*)|_{\mathbf{K}_0}$ to $\mathbf{z}$, then it also conditionally converges to $\mathbf{z}$ w.r.t. $\boldsymbol{\sigma}(\mathbf{E},\mathbf{E}^*)$.

We now prove to the converse. Let us first make an observation: if $\mathbf{F}$ is a conditionally finitely generated vector subspace of $\mathbf{E}^{**}$, then there exists a conditionally finite subset $[\mathbf{z_n^*} \colon \mathbf{n}\leq\mathbf{m}]$ of $\mathbf{S_{E^*}}$ such that for any $\mathbf{x}^{**}\in\mathbf{F}$

$$\frac{\|\mathbf{x}^{**}\|}{\mathbf{2}} \leq \mathbf{max}\,[\,|\,\mathbf{x}^{**}(\mathbf{z_n^*})\,| \colon \mathbf{n}\leq\mathbf{m}]\,. \tag{B.3}$$

Indeed, due to the conditional Heine-Borel theorem (see [65, Theorem 5.5.8]), we have that $\mathbf{S_F}$ is conditionally $\|\cdot\|$-compact in $\mathbf{F}$. Every conditional open covering has a conditionally finite open subcovering (see Definition 3.24 and Proposition 3.25 of [32]). Then there exists a conditional finite subset $[\mathbf{z_n^{**}} \colon \mathbf{1}\leq\mathbf{n}\leq\mathbf{m}]$ of $\mathbf{S_F}$ such that

$$\mathbf{S_F} \sqsubset \bigsqcup_{\mathbf{1}\leq\mathbf{n}\leq\mathbf{m}} \mathbf{B_{1/4}}(\mathbf{z_n^{**}}). \tag{B.4}$$

For each $n\in\mathbb{N}$, since $\|\,\mathbf{z^{**}_{n\wedge m}}\,\| = \mathbf{1}$, we can choose $\mathbf{z}_n^* \in \mathbf{S_{E^*}}$ so that $\mathbf{z}_n^{**}(\mathbf{z_1^*}) \geq \frac{\mathbf{3}}{\mathbf{4}}$. Now, for $\mathbf{n}\in\mathbf{N}$ with $\mathbf{n}\leq\mathbf{m}$, where $n=\sum n_i|a_i$, we define $\mathbf{z_n^*}$ with $z_n^* := \sum z_{n_i}^*|a_i$. Then, the conditional family $(\mathbf{z_n^*})_{\mathbf{1}\leq\mathbf{n}\leq\mathbf{m}}$ satisfies the condition $\mathbf{z_n^{**}}(\mathbf{z_n^*}) \geq \frac{\mathbf{3}}{\mathbf{4}}$.

Now, in view of (B.4), we have that for each $\mathbf{x}^{**}\in\mathbf{S_F}$ there is $\mathbf{n}\leq\mathbf{m}$ such that

$$\mathbf{x}^{**}(\mathbf{z_n^*}) = \mathbf{z_n^{**}}(\mathbf{z_n^*}) \,\text{-}\, (\mathbf{z_n^{**}} \,\text{-}\, \mathbf{x}^{**})(\mathbf{z_n^*}) \geq \frac{\mathbf{1}}{\mathbf{2}}. \tag{B.5}$$

For given $\mathbf{x}^{**}\in\mathbf{F}$, put $a := \operatorname{supp}(\mathbf{x}^{**})$. Then $\mathbf{x}^{**}\|\,\mathbf{x}^{**}\,\|^{-1}|a \in \mathbf{S_F}|a$, and due to (B.5), we obtain that $\mathbf{x}^{**}(\mathbf{z_n^*})|a \geq \frac{\mathbf{1}}{\mathbf{2}}\,\|\,\mathbf{x}^{**}\,\|\,|a$ for some $\mathbf{n}\leq\mathbf{m}$. Therefore, (B.3) follows.

Let $\mathbf{K}\sqsubset\mathbf{E}$ be conditionally weakly sequentially compact. Consider $\overline{\mathbf{K}} := \overline{\mathbf{j}(\mathbf{K})}^{\boldsymbol{\sigma}(\mathbf{E}^{**},\mathbf{E}^*)}$, where $\mathbf{j} : \mathbf{E}\to\mathbf{E}^{**}$ is the natural conditional embedding. By Lemma B.4.4, $\mathbf{K}$ is conditionally bounded, and therefore $\overline{\mathbf{K}}$ is conditionally bounded as well.

We show that $\overline{\mathbf{K}}$ actually lies in $\mathbf{j}(\mathbf{E})$, which will yield that $\mathbf{K}$ is conditionally weakly compact, in view of Lemma B.4.2.

Indeed, let us take $\mathbf{x}^{**}\in\overline{\mathbf{K}}$, and choose $\mathbf{x}_1^*\in\mathbf{S_{E^*}}$. Since $\mathbf{x}^{**}$ is in the conditionally weak-$*$ closure of $\mathbf{K}$, we have that there exists $\mathbf{z}_1\in\mathbf{K}$ such that

$$|(\mathbf{x}^{**} \,\text{-}\, \mathbf{j}(\mathbf{z}_1))(\mathbf{x}_1^*)\,| \leq \mathbf{1}.$$

Note that $\mathbf{span_R}[\mathbf{x}^{**}, \mathbf{x}^{**} \,\text{-}\, \mathbf{j}(\mathbf{z}_1)]$ is a conditionally finitely generated vector subspace of $\mathbf{E}^{**}$. Set $\mathbf{n}_1 := \mathbf{1}$. The previous observation provides us with a conditionally finite subset $[\mathbf{x_n^*} \colon \mathbf{n}_1 < \mathbf{n} \leq \mathbf{n}_2] \sqsubset \mathbf{S_{E^*}}$ such that for any $\mathbf{y}^{**}\in\mathbf{span_R}[\mathbf{x}^{**}, \mathbf{x}^{**} \,\text{-}\, \mathbf{j}(\mathbf{z}_1)]$,

$$\frac{\|\mathbf{y}^{**}\|}{\mathbf{2}} \leq \mathbf{max}\,[\,|\,\mathbf{y}^{**}(\mathbf{x_n})\,| \colon \mathbf{n}_1 < \mathbf{n} \leq \mathbf{n}_2] \leq \mathbf{max}\,[\,|\,\mathbf{y}^{**}(\mathbf{x_n})\,| \colon \mathbf{1} \leq \mathbf{n} \leq \mathbf{n}_2]\,.$$

Since $\mathbf{x}^{**}$ is in the conditionally weak-$*$ closure of $\mathbf{K}$ we can choose $\mathbf{z}_2 \in \mathbf{K}$ such that

$$|(\mathbf{x}^{**} - \mathbf{j}(\mathbf{z}_2))(\mathbf{x}^*_\mathbf{n})| \leq \mathbf{1/2} \quad \text{for } \mathbf{1} \leq \mathbf{n} \leq \mathbf{n}_2.$$

Now, move on to $\mathbf{span_R}[\mathbf{x}^{**}, \mathbf{x}^{**} - \mathbf{j}(\mathbf{z}_1), \mathbf{x}^{**} - \mathbf{j}(\mathbf{z}_2)]$. Since it is a conditionally finitely generated vector subspace, we can find $[\mathbf{x}^*_\mathbf{n} \colon \mathbf{n}_2 < \mathbf{n} \leq \mathbf{n}_3] \sqsubset \mathbf{S}_{\mathbf{E}^*}$ such that

$$\frac{\|\mathbf{y}^{**}\|}{\mathbf{2}} \leq \mathbf{max}\,[|\mathbf{y}^{**}(\mathbf{x_n})| \colon \mathbf{n}_2 < \mathbf{n} \leq \mathbf{n}_3] \leq \mathbf{max}\,[|\mathbf{y}^{**}(\mathbf{x_n})| \colon \mathbf{1} \leq \mathbf{n} \leq \mathbf{n}_3]$$

for any $\mathbf{y}^{**} \in \mathbf{span_R}[\mathbf{x}^{**}, \mathbf{x}^{**} - \mathbf{j}(\mathbf{z}_1), \mathbf{x}^{**} - \mathbf{j}(\mathbf{z}_2)]$.

By recurrence, we obtain two sequences, $\mathbf{n}_1 < \mathbf{n}_2, \ldots$ and $\mathbf{z}_1, \mathbf{z}_2, \ldots$, and a conditional sequence $(\mathbf{x}^*_\mathbf{n})$ such that for each $k \in \mathbb{N}$ it holds that

$$|(\mathbf{x}^{**} - \mathbf{j}(\mathbf{z}_k))(\mathbf{x}^*_\mathbf{n})| \leq \mathbf{1/k} \quad \text{for } \mathbf{1} \leq \mathbf{n} \leq \mathbf{n}_k,$$

$$\frac{\|\mathbf{y}^{**}\|}{\mathbf{2}} \leq \mathbf{max}\,[|\mathbf{y}^{**}(\mathbf{x_n})| \colon \mathbf{1} \leq \mathbf{n} \leq \mathbf{n}_k]\,,$$

for any $\mathbf{y}^{**} \in \mathbf{span_R}[\mathbf{x}^{**}, \mathbf{x}^{**} - \mathbf{j}(\mathbf{z}_1), \ldots, \mathbf{x}^{**} - \mathbf{j}(\mathbf{z}_k)]$.

Now, for $\mathbf{n} \in \mathbf{N}$ with $n := \sum n_i | a_i$ define $z_n := \sum z_{n_i} | a_i$. Then $(\mathbf{z_n})_{\mathbf{n} \in \mathbf{N}}$ is a conditional sequence in $\mathbf{K}$ which verifies that

$$\frac{\|\mathbf{y}^{**}\|}{\mathbf{2}} \leq \mathbf{sup}\,[|\mathbf{y}^{**}(\mathbf{x}^*_\mathbf{n})| \colon \mathbf{n} \in \mathbf{N}] \tag{B.6}$$

for any $\mathbf{y}^{**} \in \mathbf{span_R}[\mathbf{x}^{**}, \mathbf{x}^{**} - \mathbf{j}(\mathbf{z}_1), \mathbf{x}^{**} - \mathbf{j}(\mathbf{z}_2), \ldots]$.

Further, for $\mathbf{k} \in \mathbf{N}$ with $k := \sum k_j | b_j \in N$, define $n_k := \sum n_{k_j} | b_j \in N$. Then, for each $\mathbf{k} \in \mathbf{N}$, it holds

$$|(\mathbf{x}^{**} - \mathbf{j}(\mathbf{z_k}))(\mathbf{x}^*_\mathbf{n})| \leq \mathbf{k}^{-1} \quad \text{for } \mathbf{1} \leq \mathbf{n} \leq \mathbf{n_k}. \tag{B.7}$$

On the other hand, given that $\mathbf{K}$ is conditionally weakly sequentially compact, we have a conditional cluster point $\mathbf{x}$ of $(\mathbf{z_k})$. It follows that

$$\mathbf{x} \in \overline{\mathbf{span_R}[\mathbf{z_k} \colon \mathbf{k} \in \mathbf{N}]}^{\sigma(\mathbf{E},\mathbf{E}^*)} = \overline{\mathbf{span_R}[\mathbf{z_k} \colon \mathbf{k} \in \mathbf{N}]}^{\|\cdot\|},$$

and therefore $\mathbf{x}^{**} - \mathbf{j}(\mathbf{x}) \in \overline{\mathbf{span_R}[\mathbf{x}^{**}, \mathbf{x}^{**} - \mathbf{j}(\mathbf{z}_1), \mathbf{x}^{**} - \mathbf{j}(\mathbf{z}_2), \ldots]}^{\|\cdot\|}$.

Next, we will show that (B.6) extends to any

$$\mathbf{y}^{**} \in \overline{\mathbf{span_R}[\mathbf{x}^{**}, \mathbf{x}^{**} - \mathbf{j}(\mathbf{z}_1), \mathbf{x}^{**} - \mathbf{j}(\mathbf{z}_2), \ldots]}^{\|\cdot\|}.$$

Indeed, put $\mathbf{A} := \mathbf{span_R}[\mathbf{x}^{**}, \mathbf{x}^{**} - \mathbf{j}(\mathbf{z}_1), \mathbf{x}^{**} - \mathbf{j}(\mathbf{z}_2), \ldots]$, and suppose that we can find $\mathbf{y}^{**} \in \overline{\mathbf{A}}^{\|\cdot\|}$ so that

$$\frac{\|\mathbf{y}^{**}\|}{\mathbf{2}} | a > \mathbf{sup}\,[|\mathbf{y}^{**}(\mathbf{x}^*_\mathbf{n})| \colon \mathbf{n} \in \mathbf{N}]\, | a$$

for some $a \in \mathcal{A} \setminus \{0\}$. We can assume w.l.o.g. $a = I$. Let us choose $\mathbf{r} \in \mathbf{R}$ such that

$$\frac{\|\mathbf{y}^{**}\|}{\mathbf{2}} > \mathbf{r} > \mathbf{sup}\,[|\mathbf{y}^{**}(\mathbf{x}^*_\mathbf{n})| \colon \mathbf{n} \in \mathbf{N}]\,.$$

We can find a conditional sequence $(\mathbf{y}^{**}_\mathbf{k})$ in $\mathbf{A}$, which conditionally $\|\cdot\|$-converges to $\mathbf{y}^{**}$. Pick $\mathbf{k} \in \mathbf{N}$ such that

$$\|\mathbf{y}^{**} y^{**}_\mathbf{k}\| < (\mathbf{2r} - \|\mathbf{y}^{**}\|) \wedge (\mathbf{r} - \mathbf{sup}\,[|\mathbf{y}^{**}(\mathbf{x}^*_\mathbf{n})| \colon \mathbf{n} \in \mathbf{N}])\,.$$

Then, it is simple to show that $|\mathbf{y_k^{**}}(\mathbf{x_n})| < \mathbf{r} < \frac{\|\mathbf{y_k^{**}}\|}{\mathbf{2}}$ for any $\mathbf{n} \in \mathbf{N}$; but this contradicts (B.6).

Since (B.6) is true for every $\mathbf{y}^{**} \in \overline{\mathbf{A}}^{\|\cdot\|}$, in particular it holds for $\mathbf{x}^{**} - \mathbf{j}(\mathbf{x})$, i.e.

$$\frac{\mathbf{1}}{\mathbf{2}} \| \mathbf{x}^{**} - \mathbf{j}(\mathbf{x}) \| \leq \mathbf{sup}\,[|\, \mathbf{x}^{**}(\mathbf{x_n^*}) - \mathbf{x_n^*}(\mathbf{x})\,| : \mathbf{n} \in \mathbf{N}]\,. \tag{B.8}$$

For given $\mathbf{n} \in \mathbf{N}$, let us take $\mathbf{p}$ with $\mathbf{n} \leq \mathbf{n_p}$ and fix $\mathbf{k} \geq \mathbf{p}$. Due to (B.7), we have that

$$|\, \mathbf{x}^{**}(\mathbf{x_n^*}) - \mathbf{x_n^*}(\mathbf{x})\,| \leq |(\mathbf{x}^{**} - \mathbf{j}(\mathbf{z_k}))(\mathbf{x_n^*})| + |\mathbf{x_n^*}(\mathbf{z_k}) - \mathbf{x_n^*}(\mathbf{x})\,| \leq \mathbf{p}^{-1} + |\, \mathbf{x_n^*}(\mathbf{z_k}) - \mathbf{x_n^*}(\mathbf{x})\,|\,.$$

Since $\mathbf{x}$ is a conditional $\boldsymbol{\sigma}(\mathbf{E}, \mathbf{E}^*)$-cluster point of $(\mathbf{z_k})$, we can choose $\mathbf{k} \geq \mathbf{p}$ so that

$$|\, \mathbf{x_n^*}(\mathbf{z_k}) - \mathbf{x_n^*}(\mathbf{x})\,| \leq \frac{\mathbf{1}}{\mathbf{p}}.$$

Since $\mathbf{p} \in \mathbf{N}$ is arbitrary, we obtain that $|\, \mathbf{x}^{**}(\mathbf{x_n^*}) - \mathbf{x_n^*}(\mathbf{x})\,| = \mathbf{0}$ for all $\mathbf{n} \in \mathbf{N}$, and in view of (B.24), we conclude that $\mathbf{x}^{**} = \mathbf{j}(\mathbf{x})$, which completes the proof.

□

The next result is a simple variation of the theorem above, which will be useful later:

Corollary B.4.1. *Let $(\boldsymbol{E}, \|\cdot\|)$ be a conditionally normed spaces and $\boldsymbol{K}$ a conditionally norm bounded subset of $\boldsymbol{E}$. Let $\boldsymbol{j} : \boldsymbol{E} \to \boldsymbol{E}^{**}$ be the natural conditional embedding, and suppose that*

$$\overline{\boldsymbol{j}(\boldsymbol{K})}^{\boldsymbol{\sigma}(\boldsymbol{E}^{**}, \boldsymbol{E})} \sqcap \boldsymbol{j}(\boldsymbol{E})^{\sqsubset} \quad \textit{is on } I.$$

Then, there exists a conditional sequence $(\boldsymbol{x_n})$ in $\boldsymbol{K}$ with a conditional cluster point $\boldsymbol{x}^{**} \in \boldsymbol{E}^{**} \sqcap \boldsymbol{E}$ w.r.t. the conditional topology $\boldsymbol{\sigma}(\boldsymbol{E}^{**}, \boldsymbol{E}^*)$.

Proof. Let $\mathbf{x}^{**} \in \overline{\mathbf{j}(\mathbf{K})}^{\boldsymbol{\sigma}(\mathbf{E}^{**}, \mathbf{E})} \sqcap \mathbf{j}(\mathbf{E})^{\sqsubset}$. The proof strategy of Theorem 3.3.6 allows us to construct conditional sequences $(\mathbf{x_n})$ in $\mathbf{K}$, $(\mathbf{x_n^*})$ in $\mathbf{E}^*$, and $(\mathbf{n_k})$ in $\mathbf{N}$, which is conditionally increasing, so that

$$\frac{\mathbf{1}}{\mathbf{2}} \| \mathbf{y}^{**} \| \leq \mathbf{sup}[|\, \mathbf{y}^{**}(\mathbf{x_n^*})\,| : \mathbf{n} \in \mathbf{N}], \quad \text{for all } \mathbf{y}^{**} \in \overline{[\mathbf{x}^{**}, \mathbf{x}^{**} - \mathbf{x_n} : \mathbf{n} \in \mathbf{N}]}^{\|\cdot\|} \tag{B.9}$$

$$|\, \mathbf{x}^{**}(\mathbf{x_n^*}) - \mathbf{x_n^*}(\mathbf{x_k})\,| \leq \frac{\mathbf{1}}{\mathbf{k}} \quad \text{for all } \mathbf{1} \leq \mathbf{n} \leq \mathbf{n_k}. \tag{B.10}$$

On the other hand, the conditional Banach-Alaoglu Theorem (see Theorem B.3.1) yields that $\overline{\mathbf{j}(\mathbf{K})}^{\omega^*}$ is conditionally weakly-$*$ compact. Then, $(\mathbf{j}(\mathbf{x_n}))$ has a conditional weak-$*$ cluster point $\mathbf{x}_0^{**} \in \overline{\mathbf{j}(\mathbf{K})}^{\omega^*}$. Let us show that $\mathbf{x}_0^{**} = \mathbf{x}^{**}$.

Due to (B.9), it follows

$$\frac{\mathbf{1}}{\mathbf{2}} \| \mathbf{x}^{**} - \mathbf{x}_0^{**} \| \leq \mathbf{sup}[|\, \mathbf{x}^{**}(\mathbf{x_n^*}) - \mathbf{x}_0^{**}(\mathbf{x_n^*})\,| : \mathbf{n} \in \mathbf{N}]. \tag{B.11}$$

Now, for fixed $\mathbf{n} \in \mathbf{N}$, take $\mathbf{p}$ with $\mathbf{n} \leq \mathbf{n_p}$ and fix $\mathbf{m} \leq \mathbf{p}$. It holds

$$|\, \mathbf{x}^{**}(\mathbf{x_n^*}) - \mathbf{x}_0^{**}(\mathbf{x_n^*})\,| \leq |\, \mathbf{x}^{**}(\mathbf{x_n^*}) - \mathbf{x_n^*}(\mathbf{x_m})\,| + |\, \mathbf{x_n^*}(\mathbf{x_m})\,|\, \mathbf{x}_0^{**}(\mathbf{x_n^*})\,| \leq \frac{\mathbf{1}}{\mathbf{p}} + |\, \mathbf{x_n^*}(\mathbf{x_k}) - \mathbf{x}_0^{**}(\mathbf{x_n^*})\,|\,.$$

Given that $\mathbf{x}_0^{**}$ is a conditional ω^*-cluster point of $(\mathbf{j}(\mathbf{x_n}))$, we can choose $\mathbf{m} \geq \mathbf{p}$ so that $|\, \mathbf{x_n^*}(\mathbf{x_m}) - \mathbf{x}_0^{**}(\mathbf{x_n^*})\,| \leq \frac{\mathbf{1}}{\mathbf{p}}$. Finally, since $\mathbf{p} \in \mathbf{N}$ is arbitrary, in view of (B.11), we conclude that $\mathbf{x}_0^{**} = \mathbf{x}^{**}$. □

Let us turn to the proof of the conditional analogue of the Amir-Lindenstrauss theorem. We need some preliminary results:

Lemma B.4.6. *Let $\boldsymbol{E}$ be conditionally Banach. Suppose that $\boldsymbol{D} \sqsubset \boldsymbol{E}^*$ (resp. $\boldsymbol{D} \sqsubset \boldsymbol{E}^*$) is conditionally dense. If $\boldsymbol{K} \sqsubset \boldsymbol{E}$ (resp. $\boldsymbol{K} \sqsubset \boldsymbol{E}^*$) is conditionally norm bounded, then $\boldsymbol{\sigma}(\boldsymbol{E}, \boldsymbol{E}^*)$ and $\boldsymbol{\sigma}(\boldsymbol{E}, \boldsymbol{D})$ (resp. $\boldsymbol{\sigma}(\boldsymbol{E}^*, \boldsymbol{E})$ and $\boldsymbol{\sigma}(\boldsymbol{E}^*, \boldsymbol{D})$) agree on $\boldsymbol{K}$.*

Proof. Let us prove the case $\mathbf{K} \sqsubset \mathbf{E}^*$; the case $\mathbf{K} \sqsubset \mathbf{E}$ is analogue. It suffices to show that $\mathbf{Id} : (\mathbf{K}, \sigma(\mathbf{E}, \mathbf{E}^*)|_{\mathbf{K}}) \to (\mathbf{K}, \sigma(\mathbf{E}, \mathbf{D})|_{\mathbf{K}})$ is a conditional homeomorphism. It is clear that it is conditionally continuous. To show that the conditional inverse is continuous, take a conditional net $(\mathbf{x_i})$ in $\mathbf{K}$ such that $\boldsymbol{\sigma}(\mathbf{E}, \mathbf{D}) - \mathbf{lim_i}\, \mathbf{x_i} = \mathbf{x} \in \mathbf{K}$, and show that $\boldsymbol{\sigma}(\mathbf{E}, \mathbf{E}^*) - \mathbf{lim_i}\, \mathbf{x_i} = \mathbf{x}$ too. Given a conditionally finite subset $\mathbf{G} \sqsubset \mathbf{E}^*$ and $\mathbf{r} \in \mathbf{R}^{++}$, consider the basic conditional neighborhood $\mathbf{U_{G,r}}$. Suppose that $\mathbf{G} = [\mathbf{x_n^*} \colon \mathbf{n} \leq \mathbf{m}]$ and fix $\mathbf{s} \in \mathbf{R}^{++}$. Then for each $n \in \mathbb{N}$, let $\mathbf{y}_n^* \in \mathbf{D}$ be such that $\| \mathbf{x_n^*} - \mathbf{y}_n^* \| \leq \mathbf{s}$. For each $\mathbf{n} \in \mathbf{N}$ with $n = \sum n_i|a_i$, define $y_n^* := \sum y_n^*|a_i$ and put $\mathbf{H} := [\mathbf{y_n^*} \colon \mathbf{n} \leq \mathbf{m}]$. Then there is $\mathbf{i}'$ such that $\mathbf{x_i} \in \mathbf{x} + \mathbf{U_{H,r/2}}$ for all $\mathbf{i} \geq \mathbf{i}'$. Set $\mathbf{t} := \mathbf{sup}[\| \mathbf{x} \| \colon \mathbf{x} \in \mathbf{K}]$. Thus, for $\mathbf{i} \geq \mathbf{i}'$ and $\mathbf{n} \leq \mathbf{m}$

$$|\, \mathbf{x_n^*}(\mathbf{x_i} - \mathbf{x}) \,| \leq |(\mathbf{x_n^*} - \mathbf{y_n^*})(\mathbf{x_i}) \,| + |\, \mathbf{y_n^*}(\mathbf{x_i} - \mathbf{x}) \,| + |(\mathbf{y_n^*} - \mathbf{x_n^*})(\mathbf{x}) \,| \leq$$

$$\leq \mathbf{r/2} + \mathbf{2st},$$

We are done by just taking $\mathbf{s} := \frac{\mathbf{r}}{\mathbf{4(t+1)}}$. □

Proposition B.4.1. *The class of conditional Banach spaces having conditionally weakly-$*$ sequentially compact dual unit ball is closed under the operation of taking conditionally dense continuous linear images.*

Proof. For $\mathbf{T} \in \mathbf{B}(\mathbf{E}, \mathbf{F})$ with $\overline{\mathbf{T}(\mathbf{E})}^{\|\cdot\|} = \mathbf{F}$, it can be verified by inspection that $\mathbf{T}^* : \mathbf{F}^* \to \mathbf{E}^*$ defined by $\mathbf{T}^*(\mathbf{y}^*)(\mathbf{x}) := \mathbf{y}^*(\mathbf{T}(\mathbf{x}))$ is a conditionally bounded operator, i.e. $\mathbf{T}^* \in \mathbf{L}(\mathbf{F}^*, \mathbf{E}^*)$.

Also, it is conditionally injective. Indeed, let us suppose $\mathbf{T}^*(\mathbf{y}^*) \equiv \mathbf{0}$; that is, $\mathbf{y}^*(\mathbf{T}(\mathbf{x})) = \mathbf{0}$ for all $\mathbf{x} \in \mathbf{E}$. We show that $\mathbf{y}^* \equiv \mathbf{0}$. For a given $\mathbf{z} \in \mathbf{F}$, since $\overline{\mathbf{T}(\mathbf{E})}^{\|\cdot\|} = \mathbf{F}$, there exists a conditional sequence $(\mathbf{x_n})$ in $\mathbf{E}$ such that $\mathbf{T}(\mathbf{x_n})$ conditionally converges to $\mathbf{z}$. Then $\mathbf{y}^*(\mathbf{z}) = \mathbf{lim}\, \mathbf{y}^*(\mathbf{T}(\mathbf{x_n})) = \mathbf{0}$.

It follows that $\mathbf{T}^* : (\mathbf{B_{F^*}}, \boldsymbol{\sigma}(\mathbf{F}^*, \mathbf{T}(\mathbf{E}))|_{\mathbf{B_{F^*}}}) \to (\mathbf{T}^*(\mathbf{B_{F^*}}), \boldsymbol{\sigma}(\mathbf{E}^*, \mathbf{E})|_{\mathbf{T^*(B_{F^*})}})$ is a conditional homeomorphism. In fact, a close look shows that $\mathbf{T}^*$ defines a conditional bijection between basic conditional basic neighborhoods of $\boldsymbol{\sigma}(\mathbf{F}^*, \mathbf{T}(\mathbf{E}))|_{\mathbf{B_F^*}}$ and basic conditional basic neighborhoods of $\boldsymbol{\sigma}(\mathbf{E}^*, \mathbf{E})|_{\mathbf{T^*(B_{F^*})}}$. Since $\mathbf{T}(\mathbf{E}) \sqsubset \mathbf{F}$ is conditionally dense, due to Lemma B.4.6, $\mathbf{T}^*$ is a conditionally weak-$*$ homeomorphism between $\mathbf{B_{F^*}}$ and $\mathbf{T}^*(\mathbf{B_{F^*}})$. □

The following lemma is a conditional version of a result due to A. Grothendieck:

Lemma B.4.7. *Let $\boldsymbol{E}$ be a conditional Bananch space, and let $\boldsymbol{K} \sqsubset \boldsymbol{E}$ be on 1 and conditionally weakly closed. Suppose that we have a a conditional sequence $(\boldsymbol{K_n})$ of conditionally weakly compact subsets of $\boldsymbol{E}$ such that*

$$\boldsymbol{K} \sqsubset \boldsymbol{K}_n + \frac{\boldsymbol{1}}{\boldsymbol{2}^n} \boldsymbol{B_E} \quad \text{for all } \boldsymbol{n} \in \boldsymbol{N}.$$

Then $\boldsymbol{K}$ is conditionally $\boldsymbol{\sigma}(\boldsymbol{E}, \boldsymbol{E}^*)$-compact.

Proof. Let $\overline{\mathbf{K}}^{\omega^*}$ denote the conditional $\sigma(\mathbf{E}^{**}, \mathbf{E}^*)$-closure of $\mathbf{j}(\mathbf{K})$, with $\mathbf{j}$ the natural conditional embedding. Due to Lemma B.4.2, it suffices to show that $\overline{\mathbf{K}}^{\omega^*} \sqsubset \mathbf{E}$.

For each $\mathbf{n} \in \mathbf{N}$ it holds that $\overline{\mathbf{K_n}}^{\omega^*} = \mathbf{j}(\mathbf{K_n})$. Further, the conditional continuity of addition yields

$$\overline{\mathbf{K}}^{\omega^*} \sqsubset \overline{\mathbf{K_n} + \frac{\mathbf{1}}{\mathbf{2^n}}\mathbf{B_E}}^{\omega^*} \sqsubset \overline{\mathbf{K_n}}^{\omega^*} + \overline{\frac{\mathbf{1}}{\mathbf{2^n}}\mathbf{B_E}}^{\omega^*} \sqsubset \mathbf{j}(\mathbf{K_n}) + \frac{\mathbf{1}}{\mathbf{2^n}}\mathbf{B_{E^{**}}}.$$

Consequently

$$\overline{\mathbf{K}}^{\omega^*} \sqsubset \prod_{\mathbf{n}\in\mathbf{N}} \left(\mathbf{j}(\mathbf{K_n}) + \frac{\mathbf{1}}{\mathbf{2^n}}\mathbf{B_{E^{**}}}\right).$$

But $\prod\left(\mathbf{j}(\mathbf{K_n}) + \frac{\mathbf{1}}{\mathbf{2^n}}\mathbf{B_{E^{**}}}\right) \sqsubset \mathbf{j}(\mathbf{E})$.

Indeed, let $\mathbf{x} \in \prod\left(\mathbf{j}(\mathbf{K_n}) + \frac{\mathbf{1}}{\mathbf{2^n}}\mathbf{B_{E^{**}}}\right)$. For each $n \in \mathbb{N}$, take $\mathbf{y}_n \in \mathbf{K}_n$ and $\mathbf{u}_n \in \frac{\mathbf{1}}{\mathbf{2}^n}\mathbf{B_{E^{**}}}$. We can construct conditional sequences $(\mathbf{y_n})$ and $(\mathbf{u_n})$. Then $(\mathbf{y_n})$ is a conditional sequence in $\mathbf{j}(\mathbf{K})$ which conditionally converges to $\mathbf{x}$. Given that $\mathbf{E}$ is conditionally complete, we have that $\mathbf{j}(\mathbf{E})$ is conditionally closed in $\mathbf{E}^{**}$, and it follows that $\mathbf{x} \in \mathbf{j}(\mathbf{E})$. □

Definition B.4.3. *Let $\boldsymbol{E}$ be a conditional vector space and $\boldsymbol{C} \sqsubset \boldsymbol{E}$. The conditional gauge functional is defined by*

$$\|\,\boldsymbol{x}\,\|_{\boldsymbol{C}} := \mathbf{inf}\left[\boldsymbol{r} \in \boldsymbol{R}^{++} : \boldsymbol{x} \in \boldsymbol{rC}\right], \text{ for any } \boldsymbol{x} \in \boldsymbol{E}.$$

Then, we have the following result:

Proposition B.4.2. *Let $\boldsymbol{E}$ be a conditional vector space and $\boldsymbol{C} \sqsubset \boldsymbol{E}$, then:*

1. *if $\boldsymbol{C}$ is conditionally absorbing, then $\|\,\boldsymbol{x}\,\|_{\boldsymbol{C}} < \boldsymbol{\infty}$ for all $\boldsymbol{x} \in \boldsymbol{E}$;*
2. *if $\boldsymbol{C}$ is conditionally balanced, then $\|\,\boldsymbol{rx}\,\|_{\boldsymbol{C}} = |\,\boldsymbol{r}\,|\,\|\,\boldsymbol{x}\,\|_{\boldsymbol{C}}$ for all $(\boldsymbol{x}, \boldsymbol{r}) \in \boldsymbol{E} \times \boldsymbol{R}$;*
3. *if $\boldsymbol{C}$ is conditionally convex, then $\|\,\boldsymbol{x} + \boldsymbol{y}\,\|_{\boldsymbol{C}} \leq \|\,\boldsymbol{x}\,\|_{\boldsymbol{C}} + \|\,\boldsymbol{y}\,\|_{\boldsymbol{C}}$ for all $(\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{E} \times \boldsymbol{E}$.*

In particular, $\|\cdot\|_{\boldsymbol{C}}$ is a conditional seminorm whenever $\boldsymbol{C}$ is conditionally convex, conditionally absorbing, and conditionally balanced.

Proof. 1. Let $\mathbf{x} \in \mathbf{E}$. If $\mathbf{r} \in \mathbf{R}^{++}$ with $\mathbf{x} \in \mathbf{rC}$, then $\|\,\mathbf{x}\,\|_{\mathbf{C}} \leq \mathbf{r}$.

2. Suppose that $\mathbf{x} \in \mathbf{E}$ and $\mathbf{r} \in \mathbf{R}$. Let us put $a := \mathrm{supp}(\mathbf{r})$. First, we have that $\|\,\mathbf{rx}\,\|_{\mathbf{C}}\,|a^c = \|\,\mathbf{0}\,\|_{\mathbf{C}}\,|a^c = \mathbf{0}|a^c = \mathbf{r}\,\|\,\mathbf{x}\,\|_{\mathbf{C}}\,|a^c$.

   Let us assume $a = I$ by consistently arguing on a. Thereby, $|\,\mathbf{r}\,| > \mathbf{0}$. Besides, for fixed $\mathbf{s} \in \mathbf{R}^{++}$, being $\mathbf{C}$ conditionally balanced, we have that $\mathbf{rx} \in \mathbf{sC}$ if, and only if, $\mathbf{x} \in \frac{\mathbf{s}}{|\,\mathbf{r}\,|}\mathbf{C}$. Then, we have

$$\|\,\mathbf{rx}\,\|_{\mathbf{C}} = \mathbf{inf}\,[\mathbf{s} > \mathbf{0} \colon \mathbf{rx} \in \mathbf{sC}] = \mathbf{inf}\left[\mathbf{s} > \mathbf{0} \colon \mathbf{x} \in \frac{\mathbf{s}}{|\,\mathbf{r}\,|}\mathbf{C}\right] =$$

$$|\,\mathbf{r}\,|\,\mathbf{inf}\left[\frac{\mathbf{s}}{|\,\mathbf{r}\,|} \colon \mathbf{s} > \mathbf{0},\ \mathbf{x} \in \frac{\mathbf{s}}{|\,\mathbf{r}\,|}\mathbf{C}\right] = |\,\mathbf{r}\,|\,\|\,\mathbf{x}\,\|_{\mathbf{C}}\,.$$

3. For $\mathbf{x}, \mathbf{y} \in \mathbf{E}$, let $\mathbf{r}, \mathbf{s} \in \mathbf{R}^{++}$, with $\mathbf{x} \in \mathbf{rC}$ and $\mathbf{y} \in \mathbf{sC}$. Then, since $\mathbf{C}$ is conditionally convex, we have

$$\mathbf{x} + \mathbf{y} \in \mathbf{rC} + \mathbf{sC} = (\mathbf{r} + \mathbf{s})\left(\frac{\mathbf{r}}{\mathbf{r} + \mathbf{s}}\mathbf{C} + \frac{\mathbf{s}}{\mathbf{r} + \mathbf{s}}\mathbf{C}\right) \sqsubset (\mathbf{r} + \mathbf{s})\mathbf{C}.$$

   This implies that $\|\,\mathbf{x} + \mathbf{y}\,\|_{\mathbf{C}} \leq \mathbf{r} + \mathbf{s}$.

   Now, by taking conditional infimums; first over $\mathbf{r}$, and later over $\mathbf{s}$, we obtain the result.

□

Lemma B.4.8. *Let $\boldsymbol{E}$ be a conditionally Banach space and let $\boldsymbol{U} \sqsubset \boldsymbol{E}$ be conditionally norm mounded, convex and balanced. Let $\boldsymbol{E_U} := \mathbf{span}_{\boldsymbol{R}}\, \boldsymbol{U}$ and consider the conditional gauge function $\|\cdot\|_{\boldsymbol{U}}$, then:*

1. *$\|\cdot\|_{\boldsymbol{U}}$ is a conditional norm on $\boldsymbol{E_U}$ that conditionally induces a conditional topology which is conditionally finer that the one conditionally induced by $\|\cdot\|$ on $\boldsymbol{E_U}$;*
2. *If $\boldsymbol{U}$ is conditionally sequentially closed, then $(\boldsymbol{E_U}, \|\cdot\|)$ is conditionally Banach;*
3. *If $\boldsymbol{U}$ is a conditional neighborhood of $\boldsymbol{0} \in \boldsymbol{E}$, then $\boldsymbol{E_U} = \boldsymbol{E}$ and, in addition, $\|\cdot\|_{\boldsymbol{U}}$ and $\|\cdot\|$ conditionally induce the same topology on $\boldsymbol{E}$.*

Proof. 1. Suppose that $\mathbf{U}$ is conditionally norm bounded, convex and balanced. First, note that $\mathbf{E_U} = \sqcup_{\mathbf{n}\in\mathbf{N}} \mathbf{nU}$, hence $\mathbf{U}$ is conditionally absorbing on $\mathbf{E_U}$. Then, due to Proposition B.4.2, $\|\cdot\|_{\mathbf{U}}$ is a conditional seminorm on $\mathbf{E_U}$. Let $\mathbf{r} \in \mathbf{R}^{++}$ be such that $\|\mathbf{x}\| \leq \mathbf{r}$ for all $\mathbf{x} \in \mathbf{U}$. Fix $\mathbf{x} \in \mathbf{E_U}$ and take a conditional sequence $(\mathbf{s_n})$ in $\mathbf{R}^{++}$ with $\mathbf{x} \in \mathbf{s_n U}$ for each $\mathbf{n} \in \mathbf{N}$ which conditionally converges to $\|\mathbf{x}\|_{\mathbf{U}}$. Then $\|\mathbf{x}\| \leq \mathbf{s_n r}$ and therefore $\|\mathbf{x}\| \leq \mathbf{r}\|\mathbf{x}\|_{\mathbf{U}}$. This proves that $\|\cdot\|_{\mathbf{U}}$ is a conditional norm and that $\langle \|\cdot\| \rangle \sqsubset \langle \|\cdot\|_{\mathbf{U}} \rangle$.

2. Suppose that $\mathbf{C}$ is conditionally sequentially closed and let $(\mathbf{x_n})$ be a conditionally Cauchy sequence in $(\mathbf{E_C}, \|\cdot\|_{\mathbf{C}})$. Then, we can choose a sequence $\mathbf{n}_1 < \mathbf{n}_2 < \ldots$ such that $\mathbf{x}_{\mathbf{n}_{k+1}} - \mathbf{x}_{\mathbf{n}_k} \in \mathbf{2}^{-(\mathbf{k}+\mathbf{1})}\mathbf{C}$ for each $k \in \mathbb{N}$. For $\mathbf{k} \in \mathbf{N}$ of the form $k = \sum k_i|a_i$, we define $y_k := \sum x_{n_{k_i}}|a_i$. By doing so, we obtain a conditional sequence $(\mathbf{y_k})$ such that $\mathbf{y_{k+1}} - \mathbf{y_k} \in \mathbf{2}^{-(\mathbf{k}+\mathbf{1})}\mathbf{C}$ for each $\mathbf{k} \in \mathbf{N}$. Since $\mathbf{C}$ is conditionally norm bounded, it follows that $(\mathbf{y_k})$ is also conditionally Cauchy with respect to $\|\cdot\|$. Thus, we have that $\mathbf{lim_k}\, \mathbf{y_k} = \mathbf{x}$ for some $\mathbf{x} \in \mathbf{E}$. Now

$$\mathbf{y_n} = \mathbf{y_1} + \sum_{\mathbf{1} \leq \mathbf{k} \leq \mathbf{n}} (\mathbf{y_k} - \mathbf{y_{k-1}}) \in \mathbf{y_1} + \sum_{\mathbf{1} \leq \mathbf{k} \leq \mathbf{n}} \mathbf{2}^{-\mathbf{k}}\mathbf{C} \sqsubset \mathbf{y_1} + \mathbf{C} \quad \text{for all } \mathbf{n} \in \mathbf{N},$$

we have that $(\mathbf{y_n})$ is in $\mathbf{y_1} + \mathbf{C}$. Since $\mathbf{y_1} + \mathbf{C}$ is conditionally sequentially closed, there exists $\mathbf{x} \in \mathbf{y_1} + \mathbf{C} \sqsubset \mathbf{E_C}$ such that $(\mathbf{y_n})$ conditionally converges to $\mathbf{x}$. Let us show that $\|\mathbf{y_n} - \mathbf{x}\|_{\mathbf{C}}$ conditionally converges to $\mathbf{0}$. For $\mathbf{p} \geq \mathbf{q} \geq \mathbf{2}$, $\mathbf{p}, \mathbf{q} \in \mathbf{N}$,

$$\mathbf{y_p} - \mathbf{y_q} = \sum_{\mathbf{q}+\mathbf{1} \leq \mathbf{k} \leq \mathbf{p}} (\mathbf{y_k} - \mathbf{y_{k-1}}) \in \sum_{\mathbf{1} \leq \mathbf{k} \leq \mathbf{p}} \mathbf{2}^{-(\mathbf{k}-\mathbf{1})}\mathbf{C} \sqsubset \mathbf{2}^{-(\mathbf{q}-\mathbf{1})}\mathbf{C}.$$

We obtain that $\mathbf{y_p} - \mathbf{y_q} \in \mathbf{2}^{-(\mathbf{q}-\mathbf{1})}\mathbf{C}$ whenever $\mathbf{p}, \mathbf{q} \in \mathbf{N}$, $\mathbf{p} \geq \mathbf{q} \geq \mathbf{2}$. Since $\mathbf{C}$ is conditionally sequentially closed, it follows that $\mathbf{x} - \mathbf{y_q} \in \mathbf{2}^{-(\mathbf{q}-\mathbf{1})}\mathbf{C}$ for all $\mathbf{q}$, and therefore $\|\mathbf{x} - \mathbf{y_q}\|_{\mathbf{C}}$ conditionally converges to $\mathbf{0}$. Then, by using that $(\mathbf{x_n})$ is conditionally Cauchy, it is easy to prove that it also conditionally converges to $\mathbf{x}$.

3. Now suppose that $\mathbf{C}$ is a conditional neighborhood of $\mathbf{0} \in \mathbf{E}$. Take a conditional ball $\mathbf{rB_E}$ conditionally contained into $\mathbf{C}$. Then, $\mathbf{rx}\|\mathbf{x}\|^{-\mathbf{1}} \in \mathbf{C}$ for all $\mathbf{x} \in \mathbf{E}$, and consequently $\mathbf{r}\|\mathbf{x}\|_{\mathbf{C}} \leq \|\mathbf{x}\|$. This proves that both conditional norms are equivalent.

□

We say that a conditionally normed space $\mathbf{E}$ is conditionally reflexive if $\mathbf{j}(\mathbf{E}) = \mathbf{E}^{**}$. We have the following:

Theorem B.4.1. *A conditional Banach space $\boldsymbol{E}$ is conditionally reflexive if, and only if, the conditional unit ball $\boldsymbol{B_E}$ is conditionally weakly compact.*

In that case, the conditional dual unit ball $\boldsymbol{B_{E^*}}$ is conditionally weakly-$*$ sequentially compact.

Proof. First suppose that $\mathbf{E}$ is conditionally reflexive, then $\mathbf{B_E}$ is conditionally weakly compact as a consequence of Lemma B.4.2.

Conversely, suppose that $\mathbf{B_E}$ is conditionally weakly compact. Then it is clear that $\mathbf{j}(\mathbf{B_E})$ is conditionally $\boldsymbol{\sigma}(\mathbf{E}^{**},\mathbf{E}^*)$-compact. Therefore, $\mathbf{j}(\mathbf{B_E})$ is a conditionally $\boldsymbol{\sigma}(\mathbf{E}^{**},\mathbf{E}^*)$-closed subset of $\mathbf{B_{E^{**}}}$. But Theorem B.3.3 yields that $\mathbf{j}(\mathbf{B_E})=\mathbf{B_{E^{**}}}$.

Let us show the second part of the theorem. By the conditional Banach-Alaoglu theorem (see Theorem B.3.1), $\mathbf{B_{E^*}}$ is conditionally weakly-$*$ compact, and as $\mathbf{E}$ is conditionally reflexive, we have that $\mathbf{B_{E^*}}$ is conditionally $\boldsymbol{\sigma}(\mathbf{E}^*,\mathbf{E}^{**})$-compact. Besides, by Theorem 3.3.6, $\mathbf{B_{E^*}}$ is conditionally sequentially $\boldsymbol{\sigma}(\mathbf{E}^*,\mathbf{E}^{**})$-compact, and by using again that $\mathbf{E}$ is conditionally reflexive, we obtain that $\mathbf{B_{E^*}}$ is conditionally weak-$*$ sequentially compact. $\square$

Lemma B.4.9. *Let $(\boldsymbol{E_n})$ be a conditionally countable family of conditional Banach spaces $(\boldsymbol{E_n},\|\cdot\|_{\boldsymbol{n}})$. Consider the conditional set*

$$\oplus^2_{\boldsymbol{n}\in\boldsymbol{N}}\boldsymbol{E_n}:=\left[(\boldsymbol{x_n})\in\prod_{\boldsymbol{n}\in\boldsymbol{N}}\boldsymbol{E_n}\colon\sum_{\boldsymbol{n}\geq\boldsymbol{1}}\|\boldsymbol{x_n}\|^{\boldsymbol{2}}_{\boldsymbol{n}}\in\boldsymbol{R}\right].$$

Then we have the following:

1. $(\oplus^2_{\boldsymbol{n}\in\boldsymbol{N}}\boldsymbol{E_n},\|\cdot\|_2)$ *with* $\|\cdot\|_2:=(\sum_{\boldsymbol{n}\geq\boldsymbol{1}}\|\boldsymbol{x_n}\|^{\boldsymbol{2}})^{\frac{1}{2}}$ *is a conditional Banach space;*
2. $\left(\oplus^2_{\boldsymbol{n}\in\boldsymbol{N}}\boldsymbol{E_n}\right)^*=\oplus^2_{\boldsymbol{n}\in\boldsymbol{N}}\boldsymbol{E}^*_{\boldsymbol{n}}$*;*
3. *The conditional set*

$$\oplus^0_{\boldsymbol{n}\in\boldsymbol{N}}\boldsymbol{E}^*_{\boldsymbol{n}}:=\left[(\boldsymbol{x}^*_{\boldsymbol{n}})\in\oplus^2_{\boldsymbol{n}\in\boldsymbol{N}}\boldsymbol{E}^*_{\boldsymbol{n}}\ \exists\boldsymbol{m}\in\boldsymbol{N},\ \boldsymbol{x_n}=\boldsymbol{0}\text{ for all }\boldsymbol{n}>\boldsymbol{m}\right]$$

is conditionally dense in $\oplus^2_{\boldsymbol{n}\in\boldsymbol{N}}\boldsymbol{E}^*_{\boldsymbol{n}}$.

Proof. 1. Inspection shows that $\oplus^2_{\mathbf{n}\in\mathbf{N}}\mathbf{E_n}$ is a conditional vector space and $\|\cdot\|$ is a conditional norm. Let us prove that $\oplus^2_{\mathbf{n}\in\mathbf{N}}\mathbf{E_n}$ is conditionally complete. Let $(\mathbf{x_n})$ be a conditionally Cauchy sequence in $\oplus^2_{\mathbf{n}\in\mathbf{N}}\mathbf{E_n}$, with $\mathbf{x_n}=(\mathbf{x^m_n})_{\mathbf{m}\in\mathbf{N}}$ for each $\mathbf{n}\in\mathbf{N}$. Let $\mathbf{r}>\mathbf{0}$ and choose an $\mathbf{n_r}\in\mathbf{N}$ such that

$$\|\mathbf{x_p}-\mathbf{x_q}\|^{\mathbf{2}}_2=\sum_{\mathbf{m}\geq\mathbf{1}}\|\mathbf{x^m_p}-\mathbf{x^m_q}\|^{\mathbf{2}}_{\mathbf{m}}\leq\mathbf{r^2},\quad\text{for all }\mathbf{p}\geq\mathbf{q}\geq\mathbf{n_r}.$$

We see that, for each $\mathbf{m}$, the conditional sequence $(\mathbf{x^m_n})_{\mathbf{n}\in\mathbf{N}}$ is conditionally Cauchy and, being $\mathbf{E_m}$ conditionally Banach, it conditionally converges to some $\mathbf{x^m}\in\mathbf{E_m}$. Besides, note that the application $m\mapsto x^m$ is stable, hence $\mathbf{x}:=(\mathbf{x^n})\in\prod_{\mathbf{n}\in\mathbf{N}}\mathbf{E_n}$.

Now, for fixed $\mathbf{k}\in\mathbf{N}$ and for each $\mathbf{n}\geq\mathbf{n_r}$, we have

$$\sum_{\mathbf{1}\leq\mathbf{m}\leq\mathbf{k}}\|\mathbf{x^m}-\mathbf{x^m_n}\|^{\mathbf{2}}_{\mathbf{m}}=\mathbf{lim_j}\sum_{\mathbf{1}\leq\mathbf{m}\leq\mathbf{k}}\|\mathbf{x^m_j}-\mathbf{x^m_n}\|^{\mathbf{2}}_{\mathbf{m}}\leq\mathbf{r^2}.$$

Then, by taking conditional limits in $\mathbf{k}$, we obtain $\|\mathbf{x}-\mathbf{x_n}\|_2\leq\mathbf{r}$ for $\mathbf{n}\geq\mathbf{n_r}$. This means that, first, $\mathbf{x}-\mathbf{x_n}\in\oplus^2_{\mathbf{n}\in\mathbf{N}}\mathbf{E_n}$ for $\mathbf{n}\geq\mathbf{n_r}$, and consequently so does $\mathbf{x}$; second, since $\mathbf{r}>\mathbf{0}$ is arbitrary, it follows that $\mathbf{lim}\,\mathbf{x_n}=\mathbf{x}$.

2. Consider the conditional function $\mathbf{T}:\oplus^2_{\mathbf{n}\in\mathbf{N}}\mathbf{E}^*_{\mathbf{n}}\to\left(\oplus^2_{\mathbf{n}\in\mathbf{N}}\mathbf{E_n}\right)^*$, $\mathbf{x}^*\mapsto\mathbf{T_{x^*}}$, where

$$\mathbf{T_{x^*}}((\mathbf{x_n})):=\textstyle\sum_{\mathbf{n}\geq\mathbf{1}}\mathbf{x}^*_{\mathbf{n}}(\mathbf{x_n}),\quad\text{with }\mathbf{x}^*=\{\mathbf{x}^*_{\mathbf{n}}\}.$$

We claim that $\mathbf{T}$ is a conditional isometric isomorphism.

A plain adaptation of the Cauchy-Schwarz inequality to conditional series allows us to show that $\mathbf{T}$ is well defined and

$$\| \mathbf{T}_{\mathbf{x}^*} \| \leq \| \mathbf{x}^* \|_2 . \tag{B.12}$$

On the other hand, let $\mathbf{x}^* = (\mathbf{x}^*_{\mathbf{n}}) \in \bigoplus^2_{\mathbf{n}\in\mathbf{N}} \mathbf{E}^*_{\mathbf{n}}$.

For each $n \in \mathbb{N}$, we can choose $\mathbf{y}_n \in \mathbf{E_n}$ with $\| \mathbf{y}_n \|_{\mathbf{n}} = \mathbf{1}$. For $\mathbf{n} \in \mathbf{N}$ of the form $n = \sum n_i|a_i$, put $\mathbf{y_n} = y_n|I$ with $y_n := \sum y_{n_i}|a_i$. Doing so, we obtain a conditional sequence $(\mathbf{y_n})$ so that $\| \mathbf{y_n} \|_{\mathbf{n}} = \mathbf{1}$ for each $\mathbf{n}$.

Now, consider the conditional sequence $(\mathbf{x_n})$ where $\mathbf{x_n} := \mathbf{x}^*_{\mathbf{n}}(\mathbf{y_n})\mathbf{y_n}$ for all $\mathbf{n}$.

Note that $\| \mathbf{x_n} \|_{\mathbf{n}} = | \mathbf{x}^*_{\mathbf{n}}(\mathbf{y_n}) |$ for each $\mathbf{n}$. Consequently we have

$$\mathbf{x}^*_{\mathbf{n}}(\mathbf{x_n}) = \| \mathbf{x_n} \|^{\mathbf{2}}_{\mathbf{n}}, \qquad \text{for each } \mathbf{n} \in \mathbf{N}. \tag{B.13}$$

Now, for given $\mathbf{n}, \mathbf{m} \in \mathbf{N}$, let $\mathbf{z}^{\mathbf{m}}_{\mathbf{n}} := \mathbf{x}_{\mathbf{n}\wedge\mathbf{m}}$.

For $\mathbf{m} \in \mathbf{N}$, we have that $\mathbf{z}^{\mathbf{m}} := (\mathbf{z}^{\mathbf{m}}_{\mathbf{n}})_{\mathbf{n}\in\mathbf{N}}$ is a conditional sequence.

Then, in view of (B.13), for fixed $\mathbf{m} \in \mathbf{N}$, it follows that

$$\mathbf{T}_{\mathbf{x}^*}(\mathbf{z}^{\mathbf{m}}) = \sum_{\mathbf{1}\leq\mathbf{n}\leq\mathbf{m}} \mathbf{x}^*_{\mathbf{n}}(\mathbf{x_n}) = \sum_{\mathbf{1}\leq\mathbf{n}\leq\mathbf{m}} \| \mathbf{x_n} \|^{\mathbf{2}}_{\mathbf{n}} .$$

The inequality $| \mathbf{T}_{\mathbf{x}^*}(\mathbf{z}^{\mathbf{m}}) | \leq \| \mathbf{T}_{\mathbf{x}^*} \| \, \| \mathbf{z}^{\mathbf{m}} \|_2$ together with the above equation, implies that

$$\sum_{\mathbf{1}\leq\mathbf{n}\leq\mathbf{m}} \| \mathbf{x_n} \|^{\mathbf{2}}_{\mathbf{n}} \leq \| \mathbf{T}_{\mathbf{x}^*} \| \, \| \mathbf{z}^{\mathbf{m}} \|_2 = \| \mathbf{T}_{\mathbf{x}^*} \| \left(\sum_{\mathbf{1}\leq\mathbf{n}\leq\mathbf{m}} \| \mathbf{x_n} \|^{\mathbf{2}}_{\mathbf{n}} \right)^{\frac{\mathbf{1}}{\mathbf{2}}} .$$

It follows that

$$\| \mathbf{T}_{\mathbf{x}^*} \| \geq \left(\sum_{\mathbf{1}\leq\mathbf{n}\leq\mathbf{m}} \| \mathbf{x}^*_{\mathbf{n}} \|^{\mathbf{2}}_{\mathbf{n}} \right)^{\frac{\mathbf{1}}{\mathbf{2}}} .$$

Since $\mathbf{m} \in \mathbf{N}$ is arbitrary, we finally obtain $\| \mathbf{T}_{\mathbf{x}^*} \| \geq \| \mathbf{x}^* \|_2$.

Let us show that $\mathbf{T}$ is conditionally surjective. Indeed, suppose that $\mathbf{x}^* \in \left(\bigoplus^2_{\mathbf{n}\in\mathbf{N}} \mathbf{E_n}\right)^*$. For each $\mathbf{n} \in \mathbf{N}$, we define $\mathbf{x}^*_{\mathbf{n}} : \mathbf{E_n} \to \mathbf{R}$, $\mathbf{x}^*_{\mathbf{n}}(\mathbf{x}) := \mathbf{x}^*((\mathbf{y_m}))$ with

$$y_m := x|a + 0|a^c, \quad a := \bigvee \{b \in \mathcal{A} \colon m|b = n|b\}.$$

By doing so, we obtain

$$| \mathbf{x}^*_{\mathbf{n}}(\mathbf{x}) | = | \mathbf{x}^*((\mathbf{y_m})) | \leq \| \mathbf{x}^* \| \, \| \{\mathbf{y_m}\} \|_2 = \| \mathbf{x}^* \| \, \| \mathbf{x} \|_{\mathbf{n}} .$$

This means that $\mathbf{x}^*_{\mathbf{n}} \in \mathbf{E_n}$ for each $\mathbf{n}$. Besides, $\mathbf{T}((\mathbf{x}^*_{\mathbf{n}})) = \mathbf{x}^*$.

3. Given $\mathbf{x} = (\mathbf{x_n}) \in \bigoplus^2_{\mathbf{n}\in\mathbf{N}} \mathbf{E_n}$, for any $\mathbf{m}$ we define

$$y^m := x_n|a + 0|a^c, \quad a := \bigvee \{b \in \mathcal{A} \colon n|b \leq m|b\}.$$

Inspection shows that $(\mathbf{y}^{\mathbf{m}})$ conditionally converges to $\mathbf{x}$.

$\square$

Lemma B.4.10. *Let $(\boldsymbol{E}, \|\cdot\|)$ be a conditional Banach space, and let $\boldsymbol{K} \sqsubset \boldsymbol{E}$ be conditionally weakly compact, convex and balanced. Then there is a conditional subset $\boldsymbol{C}$ of $\boldsymbol{E}$ which is also conditionally weakly compact, convex and balanced with $\boldsymbol{K} \sqsubset \boldsymbol{C}$, and so that $(\boldsymbol{E_C}, \|\cdot\|_{\boldsymbol{C}})$ is a conditionally reflexive Banach space.*

Proof. For each $\mathbf{n} \in \mathbf{N}$ put $\mathbf{B_n} := (\mathbf{2^n K} + \mathbf{2^{-n} B_E})$. Since $\mathbf{K}$ is conditionally closed, convex and balanced, $\mathbf{B_n}$ is conditionally weakly closed, convex and balanced. Further, $\mathbf{B_n}$ is a conditional neighborhood of $\mathbf{0} \in \mathbf{E}$ since it conditionally contains $\mathbf{2^{-n} B_E}$. By Lemma B.4.8, $\mathbf{E}$ can be conditionally renormed obtaining an equivalent conditional norm $\|\cdot\|_{\mathbf{n}}$ on $\mathbf{E}$. Let $\mathbf{C} := [\mathbf{x} \in \mathbf{E} \colon \sum_{\mathbf{n} \geq \mathbf{1}} \|\mathbf{x}\|^{\mathbf{2}}_{\mathbf{n}} \leq \mathbf{1}]$.
Then

$$\mathbf{C} = \prod_{\mathbf{n} \in \mathbf{N}} \left[\mathbf{x} \in \mathbf{E} \colon \sum_{\mathbf{1} \leq \mathbf{k} \leq \mathbf{n}} \|\mathbf{x}\|^{\mathbf{2}}_{\mathbf{k}} \leq \mathbf{1} \right].$$

So $\mathbf{C}$ is conditionally norm closed, convex and balanced; hence, by Proposition B.3.1, it is also conditionally weakly closed. It can be easily verified that $\|\mathbf{x}\|_{\mathbf{C}} = \left(\sum_{\mathbf{n} \geq \mathbf{1}} \|\mathbf{x}\|^{\mathbf{2}}_{\mathbf{n}} \right)^{\mathbf{1/2}}$ for $\mathbf{x} \in \mathbf{E_C}$. Then:

1. For $\mathbf{x} \in \mathbf{K}$ it holds that $\mathbf{2^n x} \in \mathbf{2^n K}$ which is conditionally contained in $\mathbf{B_n}$, and therefore, $\|\mathbf{x}\|_{\mathbf{n}} \leq \mathbf{2^{-n}}$. Then $\|\mathbf{x}\|_{\mathbf{C}} = \sum_{\mathbf{n} \geq \mathbf{1}} \|\mathbf{x}\|^{\mathbf{2}}_{\mathbf{n}} \leq \sum_{\mathbf{n} \geq \mathbf{1}} \mathbf{2^{-2n}} =: \mathbf{r} \leq \mathbf{1}$. So $\mathbf{K} \sqsubset \mathbf{C}$.

2. $\mathbf{C}$ is conditional weakly compact in $\mathbf{E}$. This follows from Lemma B.4.7 by just noticing that $\mathbf{C}$ is conditionally weakly closed and $\mathbf{C} \sqsubset \mathbf{2^n} + \mathbf{2^{-n} B_E}$ for each $\mathbf{n} \in \mathbf{N}$.

3. The conditional topologies $\boldsymbol{\sigma}(\mathbf{E}, \mathbf{E}^*)$ and $\boldsymbol{\sigma}(\mathbf{E_C}, \mathbf{E_C^*})$ coincide on $\mathbf{C}$. Note that the conditional function $\mathbf{E_C} \to \oplus^2_{\mathbf{n} \in \mathbf{N}} \mathbf{E_n}$ which sends $\mathbf{x}$ to $(\mathbf{y_n})$ with $\mathbf{y_n} = \mathbf{x}$ for each $\mathbf{n} \in \mathbf{N}$, is a conditional isometric embedding.

   Besides, in view of Lemma B.4.9, we have that $(\oplus^2_{\mathbf{n} \in \mathbf{N}} \mathbf{E_n})^* = \oplus^2_{\mathbf{n} \in \mathbf{N}} \mathbf{E^*_n}$, and $\oplus^0_{\mathbf{n} \in \mathbf{N}} \mathbf{E^*_n}$ is a conditionally dense subset of $\oplus^2_{\mathbf{n} \in \mathbf{N}} \mathbf{E^*_n}$.

   Therefore, due to Lemma B.4.6, we have that the conditional topologies

   $$\boldsymbol{\sigma}(\oplus^2_{\mathbf{n} \in \mathbf{N}} \mathbf{E_n}, (\oplus^2_{\mathbf{n} \in \mathbf{N}} \mathbf{E_n})^*) \quad \text{and} \quad \boldsymbol{\sigma}(\oplus^2_{\mathbf{n} \in \mathbf{N}} \mathbf{E_n}, \oplus^0_{\mathbf{n} \in \mathbf{N}} \mathbf{E^*_n})$$

   agree on the conditionally bounded subset $\mathbf{C}$.

   But on $\mathbf{C}$ the conditional topology $\boldsymbol{\sigma}(\oplus^2_{\mathbf{n} \in \mathbf{N}} \mathbf{E_n}, \oplus^0_{\mathbf{n} \in \mathbf{N}} \mathbf{E^*_n})$ turns out to be the conditional weak topology $\boldsymbol{\sigma}(\mathbf{E}, \mathbf{E}^*)$.

We conclude that, as $\mathbf{C}$ is conditionally $\boldsymbol{\sigma}(\mathbf{E_C}, \mathbf{E_C^*})$-compact, $\mathbf{E_C}$ is conditionally reflexive in view of Theorem B.4.1.

□

Proof of Theorem 3.3.7:

Suppose that $\mathbf{E} = \overline{\mathbf{span_R K}}$, where $\mathbf{K} \sqsubset \mathbf{E}$, is conditionally weakly compact, convex and balanced. Let $\mathbf{C}$ be the conditionally weakly compact, convex and balanced subset provided by Lemma B.4.10. On one hand, we have that the conditional function $\mathbf{i} : \mathbf{E_C} \to \mathbf{E}$, $\mathbf{x} \mapsto \mathbf{x}$ is conditionally linear continuous with conditionally dense range. On the other hand, second part of Theorem B.4.1 tells us that, being $\mathbf{E_C}$ conditionally reflexive, it has conditionally weakly-$*$ sequentially compact dual unit balls. Hence, by Proposition B.4.1, we conclude that $\mathbf{E}$ enjoys the same property.

□

B.5 Conditional versions of James' compactness theorem

In what follows we will to prove a conditional James' compactness theorem in the non-linear setting discussed in [87]. Before stating this result we need to introduce some terminology:

Let $(\mathbf{E}, \|\cdot\|)$ be a conditionally normed space:

- Let $(\mathbf{x_n})$ and $(\mathbf{y_n})$ be conditional sequences in $\mathbf{E}$. Then, $(\mathbf{y_n})$ is said to be a *conditional convex block* of $(\mathbf{x_n})$, if there exists a sequence of conditional natural numbers $\mathbf{1} = \mathbf{n}_1 < \mathbf{n}_2 < \dots$ and a conditional sequence $(\mathbf{r_n})$ of conditional real number with $\mathbf{0} \leq \mathbf{r_n} \leq \mathbf{1}$ for all $\mathbf{n} \in \mathbf{N}$, in such a way that

$$\sum_{\mathbf{n}_k \leq \mathbf{i} < \mathbf{n}_{k+1}} \mathbf{r_i} = \mathbf{1} \quad \text{and} \quad \sum_{\mathbf{n}_k \leq \mathbf{i} < \mathbf{n}_{k+1}} \mathbf{r_i x_i} = \mathbf{y_k} \quad \text{for each } k \in \mathbb{N}. \tag{B.14}$$

- The conditional dual unit ball $\mathbf{B_{E^*}}$ is said to be conditionally weakly-$*$ *convex block compact* provided that each conditional sequence $(\mathbf{x_n})$ in $\mathbf{B_{E^*}}$ has a conditionally convex block which is weakly-$*$ convergent.
- A conditional function $\mathbf{f} : \mathbf{E} \to \overline{\mathbf{R}}$ is said to be conditionally proper if $\mathbf{f}(\mathbf{x}) > -\boldsymbol{\infty}$ and there exists $\mathbf{x} \in \mathbf{E}$ with $\mathbf{f}(\mathbf{x}) \in \mathbf{E}$.

Next, we state the main result:

Theorem B.5.1. *[Conditional unbounded version of James' Theorem] Let $(\boldsymbol{E}, \|\cdot\|)$ be a conditional Banach space such that its conditional dual unit ball $\boldsymbol{B_{E^*}}$ is conditionally ω^*-convex block compact and let $\boldsymbol{f} : \boldsymbol{E} \to \overline{\boldsymbol{R}}$ be a conditionally proper function such that*

for all $\boldsymbol{x}^* \in \boldsymbol{E}^*$, there is a $\boldsymbol{x}_0 \in \boldsymbol{E}$ so that $\boldsymbol{x}^*(\boldsymbol{x}_0) - \boldsymbol{f}(\boldsymbol{x}_0) = \mathbf{sup}\, [\boldsymbol{x}^*(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{x}) \colon \boldsymbol{x} \in \boldsymbol{E}]$,

then for every conditional real number $\boldsymbol{y} > \mathbf{inf}_{\boldsymbol{E}} \boldsymbol{f}$, the conditional sub-level set $\boldsymbol{V_y}(\boldsymbol{f}) := [\boldsymbol{z} \colon \boldsymbol{f}(\boldsymbol{z}) \leq \boldsymbol{y}]$ is conditionally weakly relatively compact.

To prove the statement above, we need some preliminaries.

Given a conditional sequence $\{\mathbf{f_n}\}$ of conditional functions $\mathbf{f_n} : \mathbf{E} \to \mathbf{R}$ defined on a conditional set $\mathbf{E}$, such that for each $\mathbf{x}$ in $\mathbf{E}$, it holds that $\mathbf{sup}_{\mathbf{n} \in \mathbf{N}} |\mathbf{f_n}(\mathbf{x})| \in \mathbf{R}$. We define

$$\mathbf{co}_{\sigma,\mathbf{R}}[\mathbf{f_n} \colon \mathbf{n} \geq \mathbf{1}] := \left[\sum_{\mathbf{n} \geq \mathbf{1}} \mathbf{r_n f_n} \colon \{\mathbf{r_n}\} \sqsubset \mathbf{R}^+, \sum_{\mathbf{n} \geq \mathbf{1}} \mathbf{r_n} = \mathbf{1} \right].$$

Notice that, due to the conditional boundedness of $\mathbf{f_n}(\mathbf{x})$, we have $\sum_{\mathbf{n} \geq 1} \mathbf{r_n f_n}(\mathbf{x}) \in \mathbf{R}$ for all $\mathbf{x}$ in $\mathbf{E}$.

Given a conditional function $\mathbf{f} : \mathbf{C} \to \mathbf{R}$, we denote the conditional supremum of $\mathbf{f}$ on $\mathbf{C}$ by $\mathbf{sup_C}\, \mathbf{f} := \mathbf{sup}[\mathbf{f}(\mathbf{x}) \colon \mathbf{x} \in \mathbf{C}]$.

Lemma B.5.1. *Let $(\boldsymbol{f_n})$ be a conditional sequence of conditional functions $\boldsymbol{f_n} : \boldsymbol{E} \to \boldsymbol{R}$ such that for each $\boldsymbol{x}$ in $\boldsymbol{E}$ it holds $\mathbf{sup}_{\boldsymbol{n}} |\boldsymbol{f_n}(\boldsymbol{x})| \in \boldsymbol{R}$ and suppose $\boldsymbol{r}$ in $\boldsymbol{R}^{++}$. Then, for every $\boldsymbol{m}$ in $\boldsymbol{N}$ there exists*

$$\boldsymbol{g_m} \in \mathbf{co}_{\sigma,\boldsymbol{R}}[\boldsymbol{f_n} \colon \boldsymbol{n} \geq \boldsymbol{m}],$$

such that

$$\mathbf{sup}_{\boldsymbol{E}} \left(\sum_{\boldsymbol{1} \leq \boldsymbol{n} \leq \boldsymbol{m}-\boldsymbol{1}} \frac{\boldsymbol{g_n}}{\boldsymbol{2^n}} \right) \leq \left(\boldsymbol{1} - \frac{\boldsymbol{1}}{\boldsymbol{2^{m-1}}} \right) \mathbf{sup}_{\boldsymbol{E}} \left(\sum_{\boldsymbol{n} \geq \boldsymbol{1}} \frac{\boldsymbol{g_n}}{\boldsymbol{2^n}} \right) + \frac{\boldsymbol{r}}{\boldsymbol{2^{m-1}}}.$$

Proof. For $\mathbf{m}$ in $\mathbf{N}$, let us define the conditional set $\mathbf{C_m} := \mathbf{co}_{\sigma,\mathbf{R}}\,[\mathbf{f_n} \colon \mathbf{n} \geq \mathbf{m}]$. In the particular case of $m \in \mathbb{N}$, it can be inductively chosen $\mathbf{g}_m$ in $\mathbf{C_m}$ satisfying

$$\gamma_m(\mathbf{g}_m) \leq \inf_{\mathbf{g}\in\mathbf{C_m}} \gamma_m(\mathbf{g}) + \frac{\mathbf{2r}}{\mathbf{4^m}} \tag{B.15}$$

with

$$\gamma_m(\mathbf{g}) := \sup_{\mathbf{x}\in\mathbf{E}} \left(\sum_{n=1}^{m-1} \frac{\mathbf{g}_n}{\mathbf{2^n}} + \frac{\mathbf{g}}{\mathbf{2^{m-1}}} \right).$$

Notice that the inductive step can be applied due to the conditional boundedness of $(\mathbf{f_n})$. Indeed, let us fix some $\mathbf{x}_0 \in \mathbf{E}$ and $\mathbf{r}_0 \in \mathbf{R}^{++}$ with $|\,\mathbf{f}(\mathbf{x}_0)\,| \leq \mathbf{r}_0$, we have that

$$\gamma_m(\mathbf{g}) \geq \left(\sum_{n=1}^{m-1} \frac{\mathbf{g}_n(\mathbf{x}_0)}{\mathbf{2^n}} + \frac{\mathbf{g}(\mathbf{x}_0)}{\mathbf{2^{m-1}}} \right) \geq -\mathbf{r}_0 \quad \text{for all } \mathbf{g} \in \mathbf{C_m},$$

thus we obtain $\inf_{\mathbf{g}\in\mathbf{C_m}} \gamma_m(\mathbf{g}) \in \mathbf{R}$ for $\mathbf{m} \in \mathbf{N}$.

For the general case of $\mathbf{m} \in \mathbf{N}$ with $m = \sum_{k\in\mathbb{N}} m_k|a_k$ with $n_k \in \mathbb{N}$ and $(a_k) \in p(I)$, we define $g_m = \sum_{k\in\mathbb{N}} g_{m_k}|a_k$. Observe that in this way $\mathbf{g_m} \in \mathbf{C_m}$ and, due to (B.15),

$$\gamma_\mathbf{m}(\mathbf{g_m}) \leq \inf_{\mathbf{g}\in\mathbf{C_m}} \gamma_\mathbf{m}(\mathbf{g}) + \frac{\mathbf{2r}}{\mathbf{4^m}} \tag{B.16}$$

with

$$\gamma_\mathbf{m}(\mathbf{g}) := \sup_{\mathbf{x}\in\mathbf{E}} \left(\sum_{\mathbf{1}\leq\mathbf{n}\leq\mathbf{m-1}} \frac{\mathbf{g_n}}{\mathbf{2^n}} + \frac{\mathbf{g}}{\mathbf{2^{m-1}}} \right).$$

for all $\mathbf{m} \in \mathbf{N}$.

Now, let us fix $m \in \mathbb{N}$. We have that

$$\mathbf{2^{m-1}} \sum_{\mathbf{n}\geq\mathbf{m}} \frac{\mathbf{g_n}}{\mathbf{2^n}} = \sum_{\mathbf{n}\geq\mathbf{m}} \sum_{\mathbf{k}\geq\mathbf{1}} \frac{\mathbf{1}}{\mathbf{2^k}} \mathbf{r_k^n f_k},$$

where, for any conditional natural number $\mathbf{n}$, we choose a conditional sequence $(\mathbf{r_k^n})$ in $\mathbf{R}^+$ with $\sum_{\mathbf{k}\geq\mathbf{1}} \mathbf{r_k^n} = \mathbf{1}$.

It can be shown that the order of summation can be interchanged for a conditional infinite series of conditionally absolutely convergent series. Hence, it holds that

$$\mathbf{2^{m-1}} \textstyle\sum_{\mathbf{n}\geq\mathbf{m}} \frac{\mathbf{g_n}}{\mathbf{2^n}} = \sum_{\mathbf{k}\geq\mathbf{1}} \mathbf{s_k f_k}, \quad \text{with } \mathbf{s_k} := \sum_{\mathbf{n}\geq\mathbf{m}} \frac{\mathbf{r_k^n}}{\mathbf{2^{n-m+1}}}$$

Since $\sum_{\mathbf{k}\geq\mathbf{1}} \mathbf{s_k} = \sum_{\mathbf{k}\geq\mathbf{1}} \mathbf{r_k} \left(\sum_{\mathbf{n}\geq\mathbf{m}} \frac{\mathbf{1}}{\mathbf{2^{n-m+1}}} \right) = \mathbf{1}$, we obtain $\mathbf{2^{m-1}} \sum_{\mathbf{n}\geq\mathbf{m}} \frac{\mathbf{g_n}}{\mathbf{2^n}} \in \mathbf{C_m}$.

So, in view of (B.16) we obtain

$$\gamma_\mathbf{m}(\mathbf{g_m}) \leq \gamma_\mathbf{m} \left(\mathbf{2^{m-1}} \sum_{\mathbf{n}\geq\mathbf{m}} \frac{\mathbf{g_n}}{\mathbf{2^n}} \right) + \frac{\mathbf{2r}}{\mathbf{4^m}} = \sup_{\mathbf{x}\in\mathbf{E}} \sum_{\mathbf{n}\geq\mathbf{1}} \frac{\mathbf{g_n}}{\mathbf{2^n}} + \frac{\mathbf{2r}}{\mathbf{4^m}}. \tag{B.17}$$

The following equation can be easily shown

$$\sum_{n=1}^{m-1} \frac{\mathbf{g_n}}{\mathbf{2^n}} = \sum_{k=1}^{m-1} \frac{\mathbf{1}}{\mathbf{2^{m-k}}} \left[\left(\sum_{n=1}^{k-1} \frac{\mathbf{g_n}}{\mathbf{2^n}} \right) + \frac{\mathbf{g_k}}{\mathbf{2^{k-1}}} \right].$$

By taking conditional supremum on $\mathbf{x} \in \mathbf{E}$ in the equality above and, by (B.17), we derive

$$\sup_{\mathbf{x}\in\mathbf{E}}\sum_{n=1}^{m-1}\frac{\mathbf{g_n}}{\mathbf{2^n}}\le\sum_{k=1}^{m-1}\frac{\mathbf{1}}{\mathbf{2^{m-k}}}\gamma_{\mathbf{k}}(\mathbf{g_k})\le\sum_{k=1}^{m-1}\frac{\mathbf{1}}{\mathbf{2^{m-k}}}\left(\sup_{\mathbf{x}\in\mathbf{E}}\sum_{\mathbf{n}\geq\mathbf{1}}\frac{\mathbf{g_n}}{\mathbf{2^n}}+\frac{\mathbf{2r}}{\mathbf{4^k}}\right)=$$

$$=\left(\mathbf{1}-\frac{\mathbf{1}}{\mathbf{2^{m-1}}}\right)\sup_{\mathbf{x}\in\mathbf{E}}\sum_{\mathbf{n}\geq\mathbf{1}}\frac{\mathbf{g_n}}{\mathbf{2^n}}+\left(\mathbf{1}-\frac{\mathbf{1}}{\mathbf{2^{m-1}}}\right)\frac{\mathbf{2r}}{\mathbf{2^m}}\le\left(\mathbf{1}-\frac{\mathbf{1}}{\mathbf{2^{m-1}}}\right)\sup_{\mathbf{x}\in\mathbf{E}}\sum_{\mathbf{n}\geq\mathbf{1}}\frac{\mathbf{g_n}}{\mathbf{2^n}}+\frac{\mathbf{r}}{\mathbf{2^{m-1}}}.$$

Finally, note that this inequality extends to any conditional natural number $\mathbf{m}$, and the proof is complete. □

The so-called Simons' inequality was established in [105, Lemma 2]. An unbounded variation was provided in [22, Theorem 2.2]. Next, we adapt the proof of the later version to the present framework.

Theorem B.5.2. *[Conditional unbounded Simons' inequality] Let $\boldsymbol{E}$ be a conditional set, let $(\boldsymbol{f_n})$ be a conditional sequence of conditional functions $\boldsymbol{f}\colon\boldsymbol{E}\to\boldsymbol{R}$ such that $\sup_{\boldsymbol{n}}|\boldsymbol{f_n}(\boldsymbol{x})|\in\boldsymbol{R}$ for each $\boldsymbol{x}\in\boldsymbol{E}$, and let $\boldsymbol{C}$ be a conditional subset of $\boldsymbol{E}$ such that for every $\boldsymbol{g}\in\mathbf{co}_{\sigma,\boldsymbol{R}}[\boldsymbol{f_n}\colon\boldsymbol{n}\geq\boldsymbol{1}]$ there exists $\boldsymbol{z}\in\boldsymbol{C}$ with $\boldsymbol{g}(\boldsymbol{z})=\sup_{\boldsymbol{E}}(\boldsymbol{g})$.*

Then,

$$\inf_{g\in\mathbf{co}_{\sigma,\boldsymbol{R}}[\boldsymbol{f_n}\colon\,\boldsymbol{n}\geq\boldsymbol{1}]}\sup{}_{\boldsymbol{E}}(\boldsymbol{g})\le\sup{}_{\boldsymbol{C}}\left(\limsup{}_{\boldsymbol{n}}\boldsymbol{f_n}\right).$$

Proof. Again, for $\mathbf{m}$ in $\mathbf{N}$, let us define the conditional set $\mathbf{C_m}:=\mathbf{co}_{\sigma,\mathbf{R}}[\mathbf{f_n}\colon\mathbf{n}\geq\mathbf{m}]$.

It suffices to prove that for every $\mathbf{r}\in\mathbf{R}^{++}$ there exist $\mathbf{z}\in\mathbf{C}$ and $\mathbf{g}\in\mathbf{C_1}$ such that $\sup_{\mathbf{E}}(\mathbf{g})-\mathbf{r}\le\limsup_{\mathbf{n}}\mathbf{f_n}(\mathbf{z})$.

Fix $\mathbf{r}$ in $\mathbf{R}^{++}$. Lemma B.5.1 provides us with a conditional sequence $(\mathbf{g_m})$ of conditional functions $\mathbf{g_m}:\mathbf{E}\to\mathbf{R}$, with $\mathbf{g_m}$ in $\mathbf{C_m}$ for all $\mathbf{m}$ in $\mathbf{N}$, such that

$$\sup{}_{\mathbf{E}}\left(\sum_{\mathbf{1}\leq\mathbf{n}\leq\mathbf{m-1}}\frac{\mathbf{g_n}}{\mathbf{2^n}}\right)\le\left(\mathbf{1}-\frac{\mathbf{1}}{\mathbf{2^{m-1}}}\right)\sup{}_{\mathbf{E}}\left(\sum_{\mathbf{n}\geq\mathbf{1}}\frac{\mathbf{g_n}}{\mathbf{2^n}}\right)+\frac{\mathbf{r}}{\mathbf{2^{m-1}}}.\tag{B.18}$$

Let us take $\mathbf{g}:=\sum_{\mathbf{n}\geq\mathbf{1}}\frac{\mathbf{g_n}}{\mathbf{2^n}}$, which is a conditional function in $\mathbf{C_1}$.

Then, by assumption, there exists $\mathbf{z}$ in $\mathbf{C}$ with $\mathbf{g}(\mathbf{z})=\sup_{\mathbf{E}}(\mathbf{g})$, and so, from (B.18) it follows that

$$\left(\mathbf{1}-\frac{\mathbf{1}}{\mathbf{2^{m-1}}}\right)\mathbf{g}(\mathbf{z})+\frac{\mathbf{r}}{\mathbf{2^{m-1}}}\geq\sup{}_{\mathbf{E}}\left(\sum_{\mathbf{1}\leq\mathbf{n}\leq\mathbf{m-1}}\frac{\mathbf{g_n}}{\mathbf{2^n}}\right)\geq\sum_{\mathbf{1}\leq\mathbf{n}\leq\mathbf{m-1}}\frac{\mathbf{g_n}(\mathbf{z})}{\mathbf{2^n}}=\mathbf{g}(\mathbf{z})-\sum_{\mathbf{n}\geq\mathbf{m}}\frac{\mathbf{g_n}(\mathbf{z})}{\mathbf{2^n}}\tag{B.19}$$

for all $\mathbf{m}$ in $\mathbf{N}$. We derive

$$\mathbf{2^{m-1}}\textstyle\sum_{\mathbf{n}\geq\mathbf{m}}\frac{\mathbf{g_n}(\mathbf{z})}{\mathbf{2^n}}\geq\mathbf{g}(\mathbf{z})-\mathbf{r}\quad\text{for all }\mathbf{m}\in\mathbf{N}.\tag{B.20}$$

We know that $\mathbf{2^{m-1}}\sum_{\mathbf{m}\leq\mathbf{n}}\frac{\mathbf{g_n}(\mathbf{z})}{\mathbf{2^n}}\in\mathbf{C_m}$ for all $\mathbf{m}\in\mathbf{N}$.

Since $\mathbf{g}(\mathbf{z})=\sup_{\mathbf{E}}(\mathbf{g})$ and in view of B.20, we therefore conclude that

$$\sup_{\mathbf{n}\geq\mathbf{m}}\mathbf{f_n}(\mathbf{z})\geq 2^{\mathbf{m-1}}\textstyle\sum_{\mathbf{n}\geq\mathbf{m}}\frac{\mathbf{g_n}(\mathbf{z})}{\mathbf{2^n}}\geq\mathbf{g}(\mathbf{z})-\mathbf{r}=\sup_{\mathbf{E}}(\mathbf{g})-\mathbf{r},\quad\text{for all }\mathbf{m}\in\mathbf{N}.$$

Now, by taking conditional infimums on $\mathbf{m}$, we obtain $\limsup_{\mathbf{n}}\mathbf{f_n}=\inf_{\mathbf{m}}\sup_{\mathbf{n}\geq\mathbf{m}}\mathbf{f_n}\geq\sup_{\mathbf{E}}(\mathbf{g})-\mathbf{r}$, as was to be shown. □

A key piece of the perturbed version of James' theorem is the so-called the sup-limsup theorem of Simons [105, Theorem 3]. Next, we adapt to the conditional setting the version provided in [22, Corollary 2.3]:

Corollary B.5.1. *[Conditional version of Simons' sup-limsup theorem] Let $\boldsymbol{E}$ be a conditional set, let $(\boldsymbol{f_n})_{n\in\boldsymbol{N}}$ be a conditional sequence of conditional functions $\boldsymbol{f_n}:\boldsymbol{E}\to\boldsymbol{R}$ such that for each $\boldsymbol{x}\in\boldsymbol{E}$ there exists $\boldsymbol{r_x}\in\boldsymbol{R}^{++}$ with $|\boldsymbol{f_n}(\boldsymbol{x})|\leq\boldsymbol{r_x}$ for all $\boldsymbol{n}\in\boldsymbol{N}$. Suppose that $\boldsymbol{C}$ is a conditional subset of $\boldsymbol{E}$ such that for every conditional function $\boldsymbol{g}$ in $\mathbf{co}_{\sigma,\boldsymbol{R}}[\boldsymbol{f_n}:\boldsymbol{n}\geq\boldsymbol{1}]$ there exists $\boldsymbol{z}\in\boldsymbol{C}$ with $\boldsymbol{g}(\boldsymbol{z})=\mathbf{sup}_{\boldsymbol{E}}(\boldsymbol{g})$. Then,*

$$\mathbf{sup}_{\boldsymbol{E}}\left(\mathbf{lim\,sup}_{\boldsymbol{n}}\boldsymbol{f_n}\right)=\mathbf{sup}_{\boldsymbol{C}}\left(\mathbf{lim\,sup}_{\boldsymbol{n}}\boldsymbol{f_n}\right).$$

Proof. We shall proceed by way of contradiction. Let us assume that there is $\mathbf{x}_0\in\mathbf{E}$ and $a\in\mathcal{A}\setminus\{0\}$, such that

$$\mathbf{lim\,sup_n\,f_n(x_0)}>\mathbf{sup_C}\left(\mathbf{lim\,sup_n\,f_n}\right)\quad \text{on } a. \tag{B.21}$$

By arguing on the Boolean algebra $\mathcal{A}_a$ if necessary, we can suppose that $a=I$ w.l.o.g.

Then, inspection shows

$$\mathbf{lim\,sup_n\,f_n(x_0)}=\inf_{\mathbf{m}}\sup_{\mathbf{n}\geq\mathbf{m}}\mathbf{f_n(x_0)}=\inf_{\mathbf{m}}\sup_{\mathbf{g}\in\mathbf{C_m}}\mathbf{g(x_0)},$$

with $\mathbf{C_m}:=\mathbf{co}_{\sigma,\mathbf{R}}[\mathbf{f_n}:\mathbf{n}\geq\mathbf{m}]$.

Now, let us fix $\mathbf{r}\in\mathbf{R}$ such that

$$\inf_{\mathbf{m}}\sup_{\mathbf{g}\in\mathbf{C_m}}\mathbf{g(x_0)}>\mathbf{r}>\mathbf{sup_C}(\lim_{\mathbf{m}}\mathbf{sup f_m})$$

Then, for each $m\in\mathbb{N}$, let us take a conditional function $\mathbf{g}_m$ in $\mathbf{C_m}$ with $\mathbf{g}_m(\mathbf{x}_0)>\mathbf{r}$. And for an arbitrary conditional natural number $\mathbf{m}$ with $m=\sum_{i\in I}m_i|a_i$, we define $\mathbf{g_m}$ as the conditional function generated by the stable function $\sum_{k\in\mathbb{N}}g_{m_k}|a_k$. Then, $\mathbf{g_m}\in\mathbf{C_m}$ for all $\mathbf{m}\in\mathbf{N}$.

By doing so, we obtain a conditional sequence $(\mathbf{g_m})$ with

$$\inf_{\mathbf{m}}\mathbf{g_m(x_0)}\geq\mathbf{r}>\mathbf{sup_C}(\lim_{\mathbf{m}}\mathbf{sup f_m})\geq\mathbf{sup_C}(\lim_{\mathbf{m}}\mathbf{sup g_m}),$$

and consequently

$$\inf_{\mathbf{g}\in\mathbf{C}_1}\mathbf{sup_E(g)}>\mathbf{sup_C}\left(\mathbf{lim\,sup_m\,g_m}\right).$$

But, on the other hand, Theorem B.5.2 tells us that

$$\inf_{\mathbf{g}\in\mathbf{C}_1}\mathbf{sup_E(g)}\leq\mathbf{sup_C}\left(\mathbf{lim\,sup_n\,g_n}\right)$$

and we have a contradiction. □

Lemma B.5.2. *Let $(\boldsymbol{E},\|\cdot\|)$ be a conditional Banach space. Suppose that the conditional dual unit ball of $\boldsymbol{E}$ is conditionally weakly-$*$ convex block compact and that $\boldsymbol{K}$ is a conditional subset of $\boldsymbol{E}$ which is conditionally norm bounded. Then $\boldsymbol{K}$ is conditionally weakly relatively compact if, and only if, each conditional sequence $(\boldsymbol{x}_n^*)$ in $\boldsymbol{E}^*$ such that $\mathbf{lim}_n\boldsymbol{x}_n^*=\boldsymbol{0}$ with $\boldsymbol{\sigma}(\boldsymbol{E}^*,\boldsymbol{E})$, also satisfies that $\mathbf{lim}_n\boldsymbol{x}_n^*=\boldsymbol{0}$ with $\boldsymbol{\sigma}(\boldsymbol{E}^*,\overline{\boldsymbol{j}(\boldsymbol{K})}^{\boldsymbol{\sigma}(\boldsymbol{E}^{**},\boldsymbol{E}^*)})$, where $\boldsymbol{j}:\boldsymbol{E}\to\boldsymbol{E}^{**}$ is the natural conditional embedding.*

Proof. Let us denote $\overline{\mathbf{K}}^{\omega^*} = \overline{\mathbf{j}(\mathbf{K})}^{\boldsymbol{\sigma}(\mathbf{E}^{**},\mathbf{E}^*)}$. If $\mathbf{K}$ is conditionally weakly relatively compact, then $\overline{\mathbf{K}}^{\omega^*} \sqsubset \mathbf{j}(\mathbf{E})$ and, in view of Lemma B.4.2, we are done.

Conversely, define

$$b := \bigvee \left\{ a \in \mathcal{A}\colon \overline{\mathbf{K}}^{\omega^*}|a \sqsubset \mathbf{j}(\mathbf{E})|a \right\}.$$

If $b = I$, we are done. If not, we can consistently argue on b^c. Thus, we can suppose $b = 0$ w.l.o.g. If so, Corollary B.4.1 guarantees the existence of a conditional sequence $(\mathbf{x_n})$ in $\mathbf{K}$ with a conditional $\boldsymbol{\sigma}(\mathbf{E}^{**},\mathbf{E}^*)$-cluster point $\mathbf{x}_0^{**} \in \mathbf{E}^{**} \sqcap \mathbf{j}(\mathbf{E})^{\sqsubset}$. We can now apply the conditional version of the separation Hahn-Banach theorem (see [32, Theorem 5.5]) to obtain $\mathbf{x}^{***} \in \mathbf{B}_{\mathbf{E}^{***}}$ such that

$$\mathbf{x}^{***}(\mathbf{x}_0^{**}) \in [\mathbf{0}]^{\sqsubset} \quad \text{and} \quad \mathbf{x}^{***}(\mathbf{j}(\mathbf{x})) = \mathbf{0} \text{ for all } \mathbf{x} \in \mathbf{E}. \tag{B.22}$$

For $\mathbf{n} \in \mathbf{N}$, let us define the conditional set

$$\mathbf{U_n} := \left[\mathbf{z}^{***} \in \mathbf{E}^{***} \ : \ \mathbf{z}^{***}(\mathbf{x}_0^{**}) \leq \frac{\mathbf{1}}{\mathbf{n}}, \mathbf{z}^{***}(\mathbf{j}(\mathbf{x_k})) \leq \frac{\mathbf{1}}{\mathbf{n}} \text{ for all } \mathbf{k} \in \mathbf{N} \text{ with } \mathbf{k} \leq \mathbf{n} \right].$$

Then $[\mathbf{U_n}\colon \mathbf{n} \in \mathbf{N}]$ is a conditional neighborhood base of $\mathbf{0} \in \mathbf{E}^{***}$ for the conditional topology $\boldsymbol{\sigma}(\mathbf{E}^{***}, [\mathbf{x}_0^{**}, \mathbf{j}(\mathbf{x_n})\colon \mathbf{n} \in \mathbf{N}])$.

Now, the conditional version of Goldstine's theorem (see Theorem B.3.3) claims that $\mathbf{j}^*(\mathbf{B}_{\mathbf{E}^*})$ is conditionally $\boldsymbol{\sigma}(\mathbf{E}^{***},\mathbf{E}^{**})$-dense in $\mathbf{B}_{\mathbf{E}^{***}}$ (where $\mathbf{j}^* : \mathbf{E}^* \to \mathbf{E}^{***}$ is the natural conditional embedding of $\mathbf{E}^*$). In particular, for each $n \in \mathbb{N}$ there exists $\mathbf{x}_n^* \in \mathbf{B}_{\mathbf{E}^*}$ with $\mathbf{j}^*(\mathbf{x}_n^*) \in \mathbf{U_n}$. For an arbitrary $\mathbf{n} \in \mathbf{N}$ with $n = \sum_i n_i|a_i$, we define $x_n^* := \sum_i x_{n_i}^*|a_i$.

Then, we obtain a conditional sequence $(\mathbf{x_n^*})$ in $\mathbf{B}_{\mathbf{E}^*}$ such that

$$\boldsymbol{\sigma}(\mathbf{E}^{***}, [\mathbf{x}_0^{**}, \mathbf{j}(\mathbf{x_n})\colon \mathbf{n} \in \mathbf{N}]) - \mathbf{lim_n}\, \mathbf{j}^*(\mathbf{x_n^*}) = \mathbf{x}^{***}. \tag{B.23}$$

Since $\mathbf{B}_{\mathbf{E}^*}$ is conditionally convex block $\boldsymbol{\sigma}(\mathbf{E}^*,\mathbf{E})$-compact, there exists a conditionally convex block compact sequence $(\mathbf{y_n^*})$ of $(\mathbf{x_n^*})$ and $\mathbf{x}_0^* \in \mathbf{B}_{\mathbf{E}^*}$ such that $\boldsymbol{\sigma}(\mathbf{E}^*,\mathbf{E}) - \mathbf{lim_n}\, \mathbf{y_n^*} = \mathbf{x}_0^*$.

Then, by assumption, we have $\boldsymbol{\sigma}(\mathbf{E}^*, \overline{\mathbf{K}}^{\omega^*}) - \mathbf{lim_n}\, \mathbf{y_n^*} = \mathbf{x}_0^*$, and so

$$\boldsymbol{\sigma}(\mathbf{E}^{***}, [\mathbf{x}_0^{**}, \mathbf{j}(\mathbf{x_n})\colon \mathbf{n} \in \mathbf{N}]) - \mathbf{lim_n}\, \mathbf{j}^*(\mathbf{y_n^*}) = \mathbf{j}^*(\mathbf{x}_0^*). \tag{B.24}$$

Finally, it follows from (B.22), (B.23) and (B.24)

$$\mathbf{x}_0^{**}(\mathbf{x}_0^*) = \mathbf{lim_n}\, \mathbf{x}_0^{**}(\mathbf{y_n^*}) = \mathbf{lim_n}\, \mathbf{x}_0^{**}(\mathbf{x_n^*}) = \mathbf{x}^{***}(\mathbf{x}_0^{**}) \in [\mathbf{0}]^{\sqsubset},$$

but for each $\mathbf{k} \in \mathbf{N}$,

$$\mathbf{x}_0^*(\mathbf{x_k}) = \mathbf{lim_n}\, \mathbf{y_n^*}(\mathbf{x_k}) = \mathbf{lim_n}\, \mathbf{x_n^*}(\mathbf{x_k}) = \mathbf{x}^{***}(\mathbf{j}(\mathbf{x_k})) = \mathbf{0},$$

which is a contradiction, because $\mathbf{x}_0^{**}$ is a conditional $\boldsymbol{\sigma}(\mathbf{E}^{**},\mathbf{E}^*)$-cluster point of $(\mathbf{x_n})$. □

Theorem B.5.3. *[Conditional and unbounded version of Rainwater–Simons' theorem] Let $(\boldsymbol{E}, \|\cdot\|)$ be a conditionally normed space and let $\boldsymbol{C}, \boldsymbol{B}$ be conditional subsets of $\boldsymbol{E}^*$ with $\boldsymbol{B} \sqsubset \boldsymbol{C}$. Suppose that $(\boldsymbol{x_n})$ is a conditionally norm bounded sequence in $\boldsymbol{E}$ such that*

$$\text{for every } \boldsymbol{x} \in \mathbf{co}_{\boldsymbol{\sigma},\boldsymbol{R}}\,[\boldsymbol{x_n}] \text{ there exists } \boldsymbol{b}^* \in \boldsymbol{B} \text{ with } \boldsymbol{b}^*(\boldsymbol{x}) = \mathbf{sup}\,[\boldsymbol{x}^*(\boldsymbol{x})\colon \boldsymbol{x}^* \in \boldsymbol{C}],$$

then,

$$\sup_{\boldsymbol{x}^* \in \boldsymbol{B}} \lim_{\boldsymbol{n}} \sup \boldsymbol{x}^*(\boldsymbol{x_n}) = \sup_{\boldsymbol{x}^* \in \boldsymbol{C}} \lim_{\boldsymbol{n}} \sup \boldsymbol{x}^*(\boldsymbol{x_n}).$$

As a consequence, if there exists a conditional sequence $(\boldsymbol{y_n})$ such that $\boldsymbol{\sigma}(\boldsymbol{E}, \boldsymbol{C})$- $\mathbf{lim}_{\boldsymbol{n}}\, \boldsymbol{y_n} = \boldsymbol{0}$ and so that

$$\textit{for every } \boldsymbol{x} \in \mathbf{co}_{\boldsymbol{\sigma},\boldsymbol{R}}\,[\boldsymbol{x_n} + \boldsymbol{y_n}] \sqcup \mathbf{co}_{\boldsymbol{\sigma},\boldsymbol{R}}\,[\text{-}\,\boldsymbol{x_n} + \boldsymbol{y_n}]$$

$$\textit{there exists } \boldsymbol{b}^* \in \boldsymbol{B} \text{ with } \boldsymbol{b}^*(\boldsymbol{x}) = \mathbf{sup}\,[\boldsymbol{x}^*(\boldsymbol{x})\colon \boldsymbol{x}^* \in \boldsymbol{C}],$$

then

$$\boldsymbol{\sigma}(\boldsymbol{E}, \boldsymbol{B})\text{-}\mathbf{lim}_n \boldsymbol{x}_n = \boldsymbol{0} \text{ implies } \boldsymbol{\sigma}(\boldsymbol{E}, \boldsymbol{C})\text{-}\mathbf{lim}_n \boldsymbol{x}_n = \boldsymbol{0}.$$

Proof. First part is a consequence of Corollary B.5.1. For the second part, fix $\mathbf{x}^* \in \mathbf{C}$, then

$$\mathbf{lim\,sup_n\,x^*(x_n) = lim\,sup_n\,x^*(x_n + y_n) \leq sup_{z^* \in C}\,lim\,sup_n\,z^*(x_n + y_n) =}$$

$$\mathbf{= sup_{z^* \in B}\,lim\,sup_n\,z^*(x_n + y_n) = lim\,sup\,z^*(x_n) = 0}.$$

On the other hand,

$$\mathbf{lim\,inf_n\,x^*(x_n) = \text{-}\,lim\,sup_n\,x^*(\text{-}\,x_n) = \text{-}\,lim\,sup_n\,x^*(\text{-}\,x_n + y_n) \geq}$$

$$\mathbf{= \text{-}\,sup_{z^* \in B}\,lim\,sup_n\,z^*(\text{-}\,x_n + y_n) = \text{-}\,lim\,sup\,z^*(\text{-}\,x_n) = 0}.$$

Then, $\mathbf{lim_n\,x^*(x_n) = 0}$, and, since $\mathbf{x}^*$ is arbitrary, we conclude that $\boldsymbol{\sigma}(\mathbf{E}, \mathbf{C})\text{-}\mathbf{lim}_{\mathbf{n}} \mathbf{x_n} = \mathbf{0}$. □

Proof of Theorem B.5.1:

Fix $\mathbf{y}_0 > \mathbf{inf_E\,f}$ and set $\mathbf{K} := \mathbf{V}_{\mathbf{y}_0}(\mathbf{f})$. Notices that $\mathbf{K}$ is on I.

The conditional uniform boundedness principle (see Theorem 3.3.5) and the optimization assumption on $\mathbf{f}$ imply that $\mathbf{K}$ is conditionally bounded. In order to obtain the conditional relative weak compactness of $\mathbf{K}$ we apply Lemma B.5.2. Thus, consider a conditional sequence $(\mathbf{x_n^*})$ in $\mathbf{E}^*$ such that $\boldsymbol{\sigma}(\mathbf{E}^*, \mathbf{E})\text{-}\mathbf{lim_n\,x_n^* = 0}$ and prove that $\boldsymbol{\sigma}(\mathbf{E}^*, \overline{\mathbf{K}}^{\omega^*})\text{-}\mathbf{lim_n\,x_n^* = 0}$.

Let us fix $(\mathbf{x}^*, \mathbf{l}) \in \mathbf{E}^* \times \mathbf{R}^{--}$. By assumption, we have $\mathbf{x}_0 \in \mathbf{E}$ such that

$$\mathbf{x^*(x_0)l^{\text{-}1} \text{-}\,f(x_0) = sup\left[x^*(x)l^{\text{-}1} \text{-}\,f(x)\colon x \in E\right]}.$$

Let us define $\mathbf{B} := \mathbf{epi(f)} \sqsubset \mathbf{C} := \overline{\mathbf{B}}^{\boldsymbol{\sigma}(\mathbf{E}^{**} \times \mathbf{R}, \mathbf{E}^* \times \mathbf{R})}$.

We claim that

$$\mathbf{sup\,[\langle(x^*, l), (x, y)\rangle\colon (x, y) \in B] = \langle(x^*, l), (x_0, f(x_0))\rangle},$$

where $\mathbf{\langle(x^*, l), (x, y)\rangle := x^*(x) + ly}$ for $\mathbf{(x^*, l, x, y) \in E^* \times R \times E \times R}$.

Indeed, define $\mathbf{z^* := x^*/l}$. Then,

$$\begin{aligned}
\mathbf{sup\,[\langle(x^*, l), (x, y)\rangle\colon (x, y) \in B\}} &\leq \mathbf{\text{-}\,l \sup_{(x,y) \in epi(g)} (z^*(x) \text{-}\,y)} \\
&\leq \mathbf{\text{-}\,l \sup_{x \in dom(f)} (z^*(x) \text{-}\,f(x))} \\
&\leq \mathbf{\text{-}\,l\sup_{x \in E} (z^*(x) \text{-}\,f(x))} \\
&= \mathbf{\text{-}\,l\,(z^*(x_0) \text{-}\,f(x_0))} \\
&= \mathbf{x^*(x_0) + lf(x_0)}.
\end{aligned}$$

Note that $\mathbf{x}_0 \in \mathbf{dom(f)}$ as $\mathbf{f}$ is conditionally proper.

Moreover, since $\mathbf{\langle(x^*, l), \cdot\rangle : E^{**} \times R \to R}$ is conditionally $\boldsymbol{\sigma}(\mathbf{E}^{**} \times \mathbf{R}, \mathbf{E}^* \times \mathbf{R})$-continuous, it holds

$$\mathbf{sup\,[\langle(x^*, l), (x, y)\rangle\colon (x, y) \in C] =}$$

$$\mathbf{= sup\,[\langle(x^*, l), (x, y)\rangle\colon (x, y) \in B] = x^*(x_0) + lf(x_0)}.$$

Now, consider the conditionally bounded sequence

$$\left(\left(\mathbf{x}^*_{\mathbf{n}}, \textbf{-}\frac{\mathbf{1}}{\mathbf{n}}\right)\right)_{\mathbf{n}\in\mathbf{N}} \text{ in } \mathbf{E}^* \times \mathbf{R}.$$

It is clear that $\boldsymbol{\sigma}(\mathbf{E}^* \times \mathbf{R}, \mathbf{B})\text{-}\mathbf{lim}_{\mathbf{n}}(\mathbf{x}^*_{\mathbf{n}}, \mathbf{0}) = \mathbf{0}$ and $\boldsymbol{\sigma}(\mathbf{E}^* \times \mathbf{R}, \mathbf{C})\text{-}\mathbf{lim}_{\mathbf{n}}(\mathbf{0}, \frac{\textbf{-1}}{\mathbf{n}}) = \mathbf{0}$. Then, in view of Theorem B.5.3, we obtain

$$\boldsymbol{\sigma}(\mathbf{E}^* \times \mathbf{R}, \mathbf{C})\text{- }\mathbf{lim}_{\mathbf{n}}(\mathbf{x}^*_{\mathbf{n}}, \mathbf{0}) = \mathbf{0},$$

and, since $\overline{\mathbf{K}}^{\omega^*} \times [\mathbf{0}] \sqsubset \mathbf{C}$, it follows that $\boldsymbol{\sigma}(\mathbf{E}^*, \overline{\mathbf{K}}^{\omega^*})\text{-}\mathbf{lim}_{\mathbf{n}}\, \mathbf{x}^*_{\mathbf{n}} = \mathbf{0}$, and the proof is complete.

□

As a consequence of Theorem B.5.1 we show a conditional analogue of the classical James' compactness Theorem —no function is involved— for conditional Banach spaces with conditionally ω^*-convex block compact dual unit ball.

Theorem B.5.4. *[Conditional version of James' Theorem] Let $(\boldsymbol{E}, \|\cdot\|)$ be a conditional Banach space such that its conditional dual unit ball $\boldsymbol{B}_{\boldsymbol{E}^*}$ is conditionally ω^*-convex block compact and let $\boldsymbol{K}$ be conditionally weakly closed subset of $\boldsymbol{E}$. The conditional set $\boldsymbol{K}$ is conditionally weakly compact if, and only if, for each $\boldsymbol{x}^* \in \boldsymbol{E}^*$ there is $\boldsymbol{x}_0 \in \boldsymbol{K}$ such that $\boldsymbol{x}^*(\boldsymbol{x}_0) = \mathbf{sup}_{\boldsymbol{x}\in\boldsymbol{K}}\, \boldsymbol{x}^*(\boldsymbol{x})$.*

Proof. Let $\mathbf{K}$ be conditionally weakly compact. For each $\mathbf{x}^* \in \mathbf{E}^*$, we know that $\mathbf{x}^*(\mathbf{K})$ is a conditionally compact subset of $\mathbf{R}$. Therefore, by the conditional version of the Heine-Borel theorem (see [65, Theorem 5.5.8]), $\mathbf{x}^*(\mathbf{K})$ is conditionally closed and bounded. From this fact, it can be showed that $\mathbf{x}^*$ attains a conditional maximum on $\mathbf{K}$.

For the converse, let us consider the conditional function $\mathbf{f}: \mathbf{E} \to \overline{\mathbf{R}}$ defined as follows: For $\mathbf{x} \in \mathbf{E}$ we define the stable function $f(x) := 1|b + \infty|b^c$, with $b := \vee\{a \in \mathcal{A}\colon x|a \in \mathbf{K}\}$. Then, $\mathbf{V_1(f)} = \mathbf{K}$ and $\mathbf{f}$ satisfies the hypothesis of Theorem B.5.1. We conclude that $\mathbf{K}$ is conditionally weakly compact. □

B.6 A Jouini-Schachermayer-Touzi theorem for conditional risk measures

In what follows we will prove Theorem 4.3.4 by means of the conditional James' compactness theorem B.5.1. Hereafter, we fix $(\Omega, \mathcal{E}, P)$ a probability space and $\mathcal{F}$ a sub-σ-algebra of $\mathcal{E}$. For $1 \leq p \leq \infty$ we consider the $L^0(\mathcal{F})$-module $L^p_{\mathcal{F}}(\mathcal{E})$. The equivalence of categories established in Theorem 3.4.1 provides us with a conditional vector space $\mathbf{L}^p := \mathbf{L}^p_{\mathcal{F}}(\mathcal{E})$.

The $L^0(\mathcal{F})$-norm $\|\cdot|\mathcal{F}\|_p : L^p_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ is a stable function, we can therefore consider the corresponding conditional function

$$\|\cdot\|_p : \mathbf{L}^p \longrightarrow \mathbf{R}$$

which is a conditional norm.

The extended conditional expectation $E_P[\cdot|\mathcal{F}] : L^1_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ is also a stable function. One can therefore consider the corresponding conditional function which is denoted by

$$\boldsymbol{E}_P : \mathbf{L}^1 \longrightarrow \mathbf{R}.$$

Suppose that $\rho : L^\infty_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ is a conditional risk measure, and consider its Fenchel conjugate

$$\rho^*(y) := \sup\{E_P[xy|\mathcal{F}] - \rho(x)\,;\, x \in L^\infty_{\mathcal{F}}(\mathcal{E})\} \quad \text{for } y \in L^1_{\mathcal{F}}(\mathcal{E}).$$

Both ρ and ρ^* are stable functions and we can consider the corresponding conditional functions

$$\boldsymbol{\rho} : \mathbf{L}^\infty \longrightarrow \mathbf{R} \quad \text{and} \quad \boldsymbol{\rho}^* : \mathbf{L}^1 \longrightarrow \overline{\mathbf{R}}.$$

Moreover, $\boldsymbol{\rho}$ is:

1. conditionally monotone, i.e. if $\mathbf{x} \leq \mathbf{y}$ then $\boldsymbol{\rho}(\mathbf{x}) \geq \boldsymbol{\rho}(\mathbf{y})$;
2. conditionally cash invariant, i.e. if $(\mathbf{r},\mathbf{x}) \in \mathbf{R}\times\mathbf{L}^p$, then $\boldsymbol{\rho}(\mathbf{x}+\mathbf{r}) = \boldsymbol{\rho}(\mathbf{x}) - \mathbf{r}$;
3. conditionally convex, i.e. $\boldsymbol{\rho}(\mathbf{r}\mathbf{x}+(\mathbf{1}-\mathbf{r})\mathbf{y}) \leq \mathbf{r}\boldsymbol{\rho}(\mathbf{x})+(\mathbf{1}-\mathbf{r})\boldsymbol{\rho}(\mathbf{y})$ for all $\mathbf{r}\in\mathbf{R}$ with $\mathbf{0}\leq\mathbf{r}\leq\mathbf{1}$ and $\mathbf{x},\mathbf{y}\in\mathbf{L}^\infty$;
4. conditionally Lipschitz continuous, i.e. $|\,\boldsymbol{\rho}(\mathbf{x}) - \boldsymbol{\rho}(\mathbf{y})\,| \leq \|\mathbf{x}-\mathbf{y}\|_\infty$, for all $\mathbf{x},\mathbf{y}\in\mathbf{L}^\infty$.

Where 1, 2, and 3 follow from the definition of conditional risk measure and Proposition 4.3.1.

Moreover, one has

$$\boldsymbol{\rho}^*(\mathbf{y}) := \mathbf{sup}\,[\boldsymbol{E}_P[\mathbf{x}\mathbf{y}] - \boldsymbol{\rho}(\mathbf{x}) : \mathbf{x}\in\mathbf{L}^\infty] \quad \text{for } \mathbf{y}\in\mathbf{L}^1.$$

We now want to introduce a conditional version for these properties. For this purpose, let us define a conditionally partial order in $\mathbf{L}^\infty$. Namely, for given $\mathbf{x},\mathbf{y}\in\mathbf{L}^\infty$ we write $\mathbf{x}\leq\mathbf{y}$ whenever if $x \leq y$. This conditional order allows to define the following notions:

- If $(\mathbf{x_n})$ is a conditional sequence in $\mathbf{L}^\infty$, we define

$$\mathbf{\liminf_n x_n} = \mathbf{\sup_m \inf_{n\geq m} x_n} \quad (\text{resp. } \mathbf{\limsup_n x_n} = \mathbf{\inf_m \sup_{n\geq m} x_n}).$$

- A conditional sequence $(\mathbf{x_n})$ in $\mathbf{L}^\infty$ is said to *conditionally converge almost surely* to $\mathbf{x}$ if $\mathbf{\liminf_n x_n} = \mathbf{\limsup_n x_n}$.
- A conditional convex risk measure $\boldsymbol{\rho} : \mathbf{L}^\infty \to \mathbf{R}$ is said to have the *conditional Fatou property* if for every conditionally bounded sequence $(\mathbf{x_n})$ in $\mathbf{L}^\infty$ which conditionally converges a.s. to $\mathbf{x}$ it holds that $\boldsymbol{\rho}(\mathbf{x}) \leq \mathbf{\liminf_n}\boldsymbol{\rho}(\mathbf{x_n})$.
- A conditional convex risk measure $\boldsymbol{\rho} : \mathbf{L}^\infty \to \mathbf{R}$ is said to have the *conditional Lebesgue property* if for every conditionally bounded sequence $(\mathbf{x_n})$ in $\mathbf{L}^\infty$ which conditionally converges a.s. to $\mathbf{x}$ it holds that $\mathbf{lim_n}\,\boldsymbol{\rho}(\mathbf{x_n}) = \boldsymbol{\rho}(\mathbf{x})$.

Remark B.6.1. *Note that no conditional topology is involved for the conditional almost sure convergence defined above.*

Let (x_n) be a sequence in $L^\infty_{\mathcal{F}}(\mathcal{E})$, and let us construct from it the conditional sequence $(\boldsymbol{x_n})$. Then, inspection shows that $\mathbf{\liminf_n}\boldsymbol{x_n} = (\liminf_n x_n)|I$ and $\mathbf{\limsup_n}\boldsymbol{x_n} = (\limsup_n x_n)|I$. This means that (x_n) converges a.s. to x if, and only if, $(\boldsymbol{x_n})$ conditionally converges a.s. to $\boldsymbol{x}$.

We have the following result:

Proposition B.6.1. *Let $\rho : L^\infty_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ be a conditional risk measure, then the following are equivalent:*

1. *$\rho|_{L^\infty(\mathcal{E})} : L^\infty(\mathcal{E}) \to L^0(\mathcal{F})$ has the Fatou property, i.e. for every bounded sequence $(x_n) \subset L^\infty(\mathcal{E})$ such that $\lim x_n = x$ a.s. it holds that $\rho(x) \leq \operatorname{ess.}\liminf_n \rho(x_n)$ (resp. Lebesgue property, i.e. for every bounded sequence $(x_n) \subset L^\infty(\mathcal{E})$ such that $\lim x_n = x$ a.s. it holds that $\rho(x) = \lim_n \rho(x_n)$);*

2. *$\boldsymbol{\rho} : \boldsymbol{L}^\infty \to \boldsymbol{R}$ has the conditional Fatou property (resp. conditional Lebesgue property).*

Proof. Let us give the proof for the Fatou property. The Lebesgue property is similar.

$2 \Rightarrow 1$: Let (x_n) be a bounded sequence of $L^\infty(\mathcal{E})$, which converges a.s. to x. It defines a conditional sequence $(\mathbf{x_n})$ in $\mathbf{L}^\infty$ which is conditionally bounded and conditionally converges a.s. to $\mathbf{x}$. Then, having $\boldsymbol{\rho}$ the conditional Fatou property it happens that $\boldsymbol{\rho}(\mathbf{x}) \leq \liminf_{\mathbf{n}} \boldsymbol{\rho}(\mathbf{x_n})$. But, it means that $\rho(x) \leq \liminf_n \rho(x_n)$.

$1 \Rightarrow 2$: Let $(\mathbf{x_n})$ be a conditionally bounded sequence which conditionally converges a.s. to $\mathbf{x}$. Then, we have that $\lim_n x_n = x$ a.s. and there exists $\eta \in L^0_+(\mathcal{F})$ with $|x_n| \leq \eta$ for every $n \in \mathbb{N}$.

For each $k \in \mathbb{N}$, let us define $a_k := \{k-1 \leq \eta < k\}$. Then $(a_k) \in p(I)$ and $|1_{a_k} x_n| \leq 1_{a_k}\eta \leq k$ for every $k \in \mathbb{N}$. Therefore, since ρ_0 has the Fatou property, we see that $1_{a_k}\rho(1_{a_k}x) \leq 1_{a_k} \liminf \rho(1_{a_k}x_n)$. Besides, ρ is stable, hence $1_{a_k}\rho(x) \leq 1_{a_k}\liminf \rho(x_n)$, which yields that $\rho(x) \leq \liminf \rho(x_n)$. □

The following result is essentially known: $1 \Leftrightarrow 2 \Leftrightarrow 4$ is proven in [55, Theorem 4.2] relying on [31, Theorem 1] and [16, Theorem 4.4]. And $4 \Leftrightarrow 5$ follows from Proposition 1.5.3 and $5 \Leftrightarrow 6$ follows from Proposition 3.4.2.

Theorem B.6.1. *Suppose that $\rho : L^\infty_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ is a conditional risk measure. Then the following conditions are equivalent:*

1. *ρ can be represented by the Fenchel conjugate ρ^*, i.e, for $x \in L^\infty_{\mathcal{F}}(\mathcal{E})$*

$$\rho(x) = \sup\left\{E_P\left[xy \mid \mathcal{F}\right] - \rho^*(y) \colon y \in L^1_{\mathcal{F}}(\mathcal{E}), y \leq 0, E\left[y \mid \mathcal{F}\right] = -1\right\}; \tag{B.25}$$

2. *$\rho|_{L^\infty(\mathcal{E})}$ has the Fatou property;*

3. *$\boldsymbol{\rho}$ has the conditional Fatou property;*

4. *the level set $V_c(\rho) := \{x \in L^\infty_{\mathcal{F}}(\mathcal{E}) \colon \rho(x) \leq c\}$ is $\sigma(L^\infty_{\mathcal{F}}(\mathcal{E}), L^1_{\mathcal{F}}(\mathcal{E}))$-closed;*

5. *the level set $V_c(\rho) := \{x \in L^\infty_{\mathcal{F}}(\mathcal{E}) \colon \rho(x) \leq c\}$ is $\sigma_{\mathfrak{s}}(L^\infty_{\mathcal{F}}(\mathcal{E}), L^1_{\mathcal{F}}(\mathcal{E}))$-closed;*

6. *the conditional level set $\boldsymbol{V_c}(\boldsymbol{\rho}) := [\boldsymbol{x} \in \boldsymbol{L}^\infty \colon \boldsymbol{\rho}(\boldsymbol{x}) \leq \boldsymbol{c}]$ is conditionally $\boldsymbol{\sigma}(\boldsymbol{L}^\infty, \boldsymbol{L}^1)$-closed.*

We will prove the following Jouini-Schachermayer-Touzi theorem for conditional risk measures, which covers Theorem 4.3.4:

Theorem B.6.2. *Suppose $\rho : L^\infty_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ is a conditional risk measure such that $\rho|_{L^\infty(\mathcal{E})}$ has the Fatou property, and put:*

$$\rho_0^*(y) := \sup\{E_P[xy|\mathcal{F}] - \rho_0(x) \colon x \in L^\infty(\mathcal{E})\} \quad \text{for } y \in L^1(\mathcal{E})$$

with $\rho_0 := \rho|_{L^\infty_{\mathcal{F}}(\mathcal{E})}$. The following statements are equivalent:

1. *The conditional set $\boldsymbol{V_c}(\boldsymbol{\rho}^*) := \left[\boldsymbol{y} \in \boldsymbol{L}^1 \colon \boldsymbol{\rho}^*(\boldsymbol{y}) \leq \boldsymbol{c}\right]$ is conditionally $\boldsymbol{\sigma}\left(\boldsymbol{L}^1, \boldsymbol{L}^\infty\right)$-compact for all $\boldsymbol{c} \in \boldsymbol{R}$;*

2. *ρ_0 has the Lebesgue property;*

3. *$\boldsymbol{\rho}$ has the conditional Lebesgue property;*

4. *for every $x \in L^\infty(\mathcal{E})$ there is $y \in L^1(\mathcal{E})$ with $y \leq 0$ and $E_P[y|\mathcal{F}] = -1$ such that $\rho(x) = E_P[xy|\mathcal{F}] - \rho_0^*(y)$;*

5. *For every $x \in L^\infty_{\mathcal{F}}(\mathcal{E})$ there is $y \in L^1_{\mathcal{F}}(\mathcal{E})$ with $y \leq 0$ and $E_P[y|\mathcal{F}] = -1$ such that $\rho(x) = E_P[xy|\mathcal{F}] - \rho^*(y)$.*

Before proving the result above, we need some preliminary results.

Proposition B.6.2. *The conditional set $\boldsymbol{L}^\infty$ is conditionally dense in $\boldsymbol{L}^1$ with respect the conditional norm topology.*

Proof. Indeed, let us choose $\mathbf{x} \in \mathbf{L}^1$. Since $x = x^+ - x^-$, we can assume w.l.o.g. that $x \geq 0$. Let us choose some sequence (x_n) in $L^\infty(\mathcal{E})$ such that $x_n \nearrow x$ a.s. Then, by monotone convergence we have $E_P[x_n - x|\mathcal{F}] \nearrow 0$. If we construct the conditional sequence $(\mathbf{x_n}) \sqsubset \mathbf{L}^\infty$, it follows that $\mathbf{lim_n} \| \mathbf{x_n} - \mathbf{x} \|_1 = \mathbf{0}$. □

Lemma B.6.1. *The conditional unit ball of $\boldsymbol{L}^\infty$ is conditionally weakly-$*$ sequentially compact*

Proof. In view of Proposition B.6.2, we have that $\mathbf{L}^2$ is conditionally dense in $\mathbf{L}^1$. This mean that $\mathbf{L}^1 = \overline{\mathbf{L}^2} = \overline{\mathbf{span_R B_{L^2}}}$, where $\mathbf{B_{L^2}}$ is the conditional unit ball of $\mathbf{L}^2$. By the conditional version of Amir-Lindenstrauss (Theorem 3.3.7), it follows that the conditional unit ball of $(\mathbf{L}^1)^* = \mathbf{L}^\infty$ is conditionally weakly-$*$ sequentially compact. □

The following results is a plain adaptation of results from [16] to the present setting (see Lemma 2.16, Proposition 3.3 and Proposition 3.5).

Lemma B.6.2. *The following properties hold:*

1. *$s(L^\infty(\mathcal{E})) = L^\infty_{\mathcal{F}}(\mathcal{E})$.*

2. *Let $\boldsymbol{\rho} : \boldsymbol{L}^\infty \to \boldsymbol{R}$ be a conditional convex risk measure. Then, for $y \in L^1(\mathcal{E})$*

$$\rho^*(y) = \sup \{E_P[xy|\mathcal{F}] - \rho(x) \colon x \in L^\infty(\mathcal{E})\}.$$

Now, let us turn to prove the main theorem.

Proof. We will follow the line $3 \Leftrightarrow 2 \Leftrightarrow 4 \Leftrightarrow 5 \Leftrightarrow 1$.

$3 \Leftrightarrow 2$ is just Proposition B.6.1.

$2 \Rightarrow 4$: We shall use a method similar to the scalarization technique used by Deflefsen and Scandolo in the proof of $a \Rightarrow c$ in [31, Theorem 1]. First, we can assume, by applying a translation if necessary, that $\rho(0) = 0$. If so, for $x \in L^\infty(\mathcal{E})$, due to Proposition 4.3.1 we have that $|\rho(x)| \leq \|x|\mathcal{F}\|_\infty \leq \|x\|_\infty$, hence $\rho(L^\infty(\mathcal{E})) \subset L^\infty(\mathcal{F})$.

Let us fix $x \in L^\infty(\mathcal{E})$ and put $\rho_0 = \rho|_{L^\infty(\mathcal{E})}$. Since $\rho(x) \geq E_P[xy|\mathcal{F}] - \rho_0^*(y)$ for all $y \in L^1(\mathcal{E})$, it suffices to show that there exists $y \in L^1(\mathcal{E})$, with $y \leq 0$ and $E_P[y|\mathcal{F}] = -1$, such that

$$E_P[\rho(x)] = E_P[E_P[xy|\mathcal{F}] - \rho_0^*(y)]. \tag{B.26}$$

Let us take $\rho' : L^\infty(\mathcal{E}) \to \mathbb{R}$, where $\rho'(x) := E_P[\rho(x)]$ for each $x \in L^\infty(\mathcal{E})$, which is a convex risk measure. Moreover, we claim that ρ' has the Lebesgue property. Indeed, given a bounded sequence $(x_n) \subset L^\infty(\mathcal{E})$ which converges a.s. to x, having ρ_0 the Lebesgue property, it holds that $\lim_n \rho_0(x_n) = \rho_0(x)$. Further, due to Proposition 4.3.1, we have that $(\rho_0(x_n))$ is bounded. Thus, by dominated convergence, we conclude that $\lim_n \rho'(x_n) = \rho'(x)$. Now, by the original Jouini-Schachermayer-Touzi Theorem, it follows that $\rho'(x) = E_P[xy] - (\rho')^*(y)$ for some $y \in L^1(\mathcal{E})$ with $y \leq 0$ and $E_P[y] = -1$, where $(\rho')^*(y) = \sup\{E_P[xy] - \rho'(x) \colon x \in L^\infty(\mathcal{E})\}$.

Furthermore, we claim that $E_P[y|\mathcal{F}] = -1$. Arguing by way of contradiction, suppose for instance that $a := \{E_P[y|\mathcal{F}] > -1\}$ has positive probability. If so, for fixed $\lambda \in \mathbb{R}^+$, we have

$$(\rho')^*(y) \geq E_P[1_a \lambda y] - \rho'(\lambda 1_a) = \lambda\left(E_P[1_a y] + P(a)\right) = \lambda E_P\left[1_a\left(E_P[y|\mathcal{F}] + 1\right)\right].$$

But this is impossible, since λ is arbitrarily large and $(\rho')^*(y) < +\infty$.

In addition, by using the same techniques as in [31, Theorem 1], we can obtain that $E_P[\rho_0^*(y)] = (\rho')^*(y)$. Therefore, we finally have

$$E_P[E_P[xy|\mathcal{F}] - \rho_0^*(y)] = E_P[xy] - (\rho')^*(y) = \rho'(y) = E_P[\rho(x)].$$

$4 \Rightarrow 2$: We will use the same reduction trick again. Let (x_n) be a bounded sequence in $L^\infty(\mathcal{E})$ such that x_n converges a.s. to x. Since ρ has the Fatou property, we have that $\rho(x) \leq \liminf_n \rho(x_n)$. It suffices to show that $\rho(x) \geq \limsup_n \rho(x_n)$. Let us argue to get contradiction. Suppose that there exists $a \in \mathcal{A}$, $a \neq 0$ such that $\rho(x) < \limsup_n \rho(x_n)$ on a. We can assume $a = I$ w.l.o.g. Then, let us consider $\rho'(x) := E_P[\rho(x)]$ for $x \in L^\infty(\mathcal{E})$. Thereby, we have

$$\rho'(x) < E_P[\limsup_n \rho(x_n)].$$

Due to Proposition 4.3.1, we can construct a bounded sequence (z_n) in a such a way that $\lim_n \rho(z_n) = \limsup_n \rho(x_n)$, and which converges a.s to x. Then, by dominated convergence, we obtain

$$\lim_n E_P[\rho(z_n)] = E_P[\lim_n \rho(z_n)] = E_P[\limsup_n \rho(x_n)] > \rho'(x). \tag{B.27}$$

But, on the other hand, by assumption we have that, for each $x \in L^\infty(\mathcal{E})$, there is $y \in L^1(\mathcal{E})$ with $y \leq 0$ and $E_P[y|\mathcal{F}] = -1$, such that $\rho'(x) = E_P[E_P[xy|\mathcal{F}] - \rho_0^*(y)] = E_P[xy] - E_P[\rho_0^*(y)] = E_P[xy] - (\rho')^*(y)$. Also, by dominated convergence, we obtain ρ' has the Fatou property. In view of the original Jouini-Schachermayer-Touzi Theorem, it follows that ρ' has the Lebesgue property. Hence, $\lim_n E_P[\rho(z_n)] = \rho'(x)$, which is a contradiction, in view of (B.27).

$4 \Rightarrow 5$: By 2 of Lemma B.6.2, it holds that $\rho^*(y) = \rho_0^*(y)$ for all $y \in L^1(\mathcal{E})$. Now, let us fix $x \in L^\infty_{\mathcal{F}}(\mathcal{E})$. Due to 1 of Lemma B.6.2, with have that $x = \sum 1_{a_k} x_k$ with $x_k \in L^\infty(\mathcal{E})$ and $\{a_k\} \in p(1)$. By assumption, we can choose $y \in L^1(\mathcal{E})$ satisfying 4. Thereby, by using the fact that ρ is stable, we have

$$\rho(x) = \sum 1_{a_k} \rho(x_k) = \sum 1_{a_k}\left(E_P[x_k y|\mathcal{F}] - \rho_0^*(y)\right) = E_P[xy|\mathcal{F}] - \rho^*(y).$$

$5 \Rightarrow 4$: For given $x \in L^\infty(\mathcal{E})$, by assumption there exists $y \in L^1_{\mathcal{F}}(\mathcal{E})$ satisfying 5. In particular, $E_P[y|\mathcal{F}] = -1$. Then $E_P[|y|] = E_P[E_P[|y||\mathcal{F}]] = 1$, hence $y \in L^1(\mathcal{E})$.

$5 \Rightarrow 1$: Due to Lemma B.6.1, we know that the conditional unit ball of $\mathbf{L}^\infty$ is conditionally weakly-$*$ sequentially compact, in particular, it is conditionally weakly convex block compact. Then, Theorem B.5.1 tells us that $\mathbf{V_c}(\boldsymbol{\rho}^*)$ is conditionally weakly compact.

$1 \Rightarrow 5$: Let us fix $\mathbf{x} \in \mathbf{L}^\infty$. Since $\rho|_{L^\infty(\mathcal{E})}$ has the Fatou property, due to Theorem B.6.1, we have that

$$\boldsymbol{\rho}(\mathbf{x}) = \mathbf{sup}\left[\boldsymbol{E}_P[\mathbf{xy}] \,\textbf{-}\, \boldsymbol{\rho}^*(\mathbf{y}) \colon \mathbf{y} \leq \mathbf{0},\ \boldsymbol{E}_P[\mathbf{y}] = \textbf{-}\,\mathbf{1}\right].$$

Thus, we can take a conditional sequence $(\mathbf{y_n})$ in $\mathbf{L}^1$ with $\mathbf{y_n} \leq \mathbf{0}$ and $\boldsymbol{E}\,[\mathbf{y_n}] = \textbf{-}\,\mathbf{1}$ for each $\mathbf{n} \in \mathbf{N}$, such that $\boldsymbol{\rho}(\mathbf{x}) = \mathbf{lim}(E_P[\mathbf{x}\mathbf{y_n}] \,\textbf{-}\, \boldsymbol{\rho}^*(\mathbf{y_n}))$. This means that the conditional sequence $(\boldsymbol{\rho}^*(\mathbf{y_n}))$ is conditionally bounded, and we therefore have $(\mathbf{y_n}) \sqsubset \mathbf{V_c}(\boldsymbol{\rho}^*)$, for some $\mathbf{c} \in \mathbf{R}^+$. Then, the conditional weak compactness and the conditional Eberlein-Šmulian theorem (see

Theorem 3.3.6) allows us to suppose that $(\mathbf{y_n})$ conditionally weakly converges to some $\mathbf{y} \in \mathbf{L}^1$ with $\mathbf{y} \leq \mathbf{0}$ and $\boldsymbol{E}[\mathbf{y}] = -\mathbf{1}$. Consequently, we have $\liminf_{\mathbf{n}} \boldsymbol{\rho}^*(\mathbf{y_n}) \geq \boldsymbol{\rho}^*(\mathbf{y})$. Thus, we finally have $\boldsymbol{\rho}(\mathbf{x}) = \lim(\boldsymbol{E}_P[\mathbf{xy}] - \boldsymbol{\rho}^*(\mathbf{y_n})) = \boldsymbol{E}_P[\mathbf{xy}] - \liminf_{\mathbf{n}} \boldsymbol{\rho}^*(\mathbf{y_n}) \leq \boldsymbol{E}_P[\mathbf{xy}] - \boldsymbol{\rho}^*(\mathbf{y})$, which means that $\boldsymbol{\rho}(\mathbf{x}) = \boldsymbol{E}_P[\mathbf{xy}] - \boldsymbol{\rho}^*(\mathbf{x})$. $\square$

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Alphabetical Index