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L_1-2-TYPE SURFACES IN 3-DIMENSIONAL DE SITTER AND ANTI DE SITTER SPACES

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ABSTRACT. Let M_s^2 be an orientable surface immersed in the De Sitter space $\mathbb{S}_1^3 \subset \mathbb{R}_1^4$ or anti de Sitter space $\mathbb{H}_1^3 \subset \mathbb{R}_2^4$. In the case that M_s^2 is of L_1-2-type we prove that the following conditions are equivalent to each other: M_s^2 has a constant principal curvature; M_s^2 has constant mean curvature; M_s^2 has constant second mean curvature. As a consequence, we also show that an L_1-2-type surface is either an open portion of a standard pseudo-Riemannian product, or a B-scroll over a null curve, or else its mean curvature, its Gaussian curvature and its principal curvatures are all non-constant.

1. INTRODUCTION

The submanifolds of finite type in the Euclidean or pseudo-Euclidean space are submanifolds whose isometric immersion in the ambient space is constructed by using eigenfunctions of their Laplacian. This notion provides a very natural way to apply spectral geometry to the study of submanifolds (see [2, 3]). This concept, originally defined for the Laplacian operator Δ, can be generalized in a natural way to other operators such as the differential operator $\Box$ (sometimes denoted by L_1) introduced by Cheng and Yau in [5] for the study of surfaces with constant scalar curvature.

Let $\mathbb{M}_c^3$ denote the 3-dimensional De Sitter or anti De Sitter space, isometrically immersed in a pseudo-Euclidean space $\mathbb{R}_q^4$ of index q. An orientable surface $\psi : M_s^2 \to \mathbb{M}_c^3 \subset \mathbb{R}_q^4$ is said to be of L_1-2-type if the position vector ψ admits the following spectral decomposition

$$\psi = a + \psi_1 + \psi_2, \quad L_1\psi_1 = \lambda_1\psi_1, \quad L_1\psi_2 = \lambda_2\psi_2, \quad \lambda_1 \neq \lambda_2, \quad \lambda_i \in \mathbb{R},$$

where a is a constant vector in $\mathbb{R}_q^4$, and ψ_1, ψ_2 are $\mathbb{R}_q^4$-valued non-constant differentiable functions on M_s^2.

The second and third author in [10, 11] studied these kind of surfaces in the non-flat Riemannian space forms. It should be noted that the shape operator in the Riemannian case is always diagonalizable so that neither the techniques used nor the results obtained can be directly transferred to the Lorentzian case, since in this case the shape operator may be non-diagonalizable.

The main theorems of this paper are the following.

Theorem A. *Let $\psi : M_s^2 \to \mathbb{M}_c^3 \subset \mathbb{R}_q^4$ be an orientable surface of L_1-2-type. The following conditions are equivalent to each other:*

1) *M_s^2 has a constant principal curvature.*
2) *H is constant.*
3) *H_2 is constant.*

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As a consequence of this theorem, the known examples of L_1-2-type surfaces in $\mathbb{M}_c^3$ (see examples 1–4) and the classification of isoparametric surfaces in $\mathbb{M}_c^3$ (see e.g. [1, 14, 15]), we have the following characterization results of L_1-2-type surfaces in $\mathbb{M}_c^3$.

Theorem B. *Let $\psi : M_s^2 \to \mathbb{S}_1^3 \subset \mathbb{R}_1^4$ be an orientable surface of L_1-2-type. Then one of the following conditions is satisfied*

- *M_s^2 an open portion of a standard pseudo-Riemannian product: $\mathbb{S}_1^1(\sqrt{1-r^2}) \times \mathbb{S}^1(r)$ or $\mathbb{H}^1(\sqrt{1-r^2}) \times \mathbb{S}^1(r)$.*
- *M_1^2 is a B-scroll over a null curve.*
- *M_s^2 has non constant mean curvature, non constant Gaussian curvature, and non constant principal curvatures.*

Theorem C. *Let $\psi : M_s^2 \to \mathbb{H}_1^3 \subset \mathbb{R}_2^4$ be an orientable surface of L_1-2-type. Then one of the following conditions is satisfied*

- *M_s^2 is an open portion of a standard pseudo-Riemannian product: $\mathbb{S}_1^1(r) \times \mathbb{H}^1(-\sqrt{r^2+1})$, or $\mathbb{H}^1(-r) \times \mathbb{H}^1(-\sqrt{1-r^2})$, or $\mathbb{H}_1^1(-r) \times \mathbb{S}^1(\sqrt{r^2-1})$*
- *M_1^2 is a non-flat B-scroll over a null curve.*
- *M_s^2 has non constant mean curvature, non constant Gaussian curvature, and non constant principal curvatures.*

2. Preliminaries

In this section we recall some formulae and notions about surfaces in Lorentzian space forms that will be used later on. Let $\mathbb{R}_q^4$ be the 4-dimensional pseudo-Euclidean space of index $q \in \{1, 2, 3\}$, with flat metric given by

$$\langle \cdot, \cdot \rangle = -\sum_{i=1}^{q} dx_i^2 + \sum_{j=q+1}^{4} dx_j^2,$$

where $x = (x_1, x_2, x_3, x_4)$ denotes the usual rectangular coordinates in $\mathbb{R}^4$. The De Sitter space of radius r is defined by

$$\mathbb{S}_1^3(r) = \big\{ x \in \mathbb{R}_1^4 \ : \ \langle x, x \rangle = r^2 \big\},$$

and the anti De Sitter space of radius $-r$ is defined by

$$\mathbb{H}_1^3(-r) = \big\{ x \in \mathbb{R}_2^4 \ : \ \langle x, x \rangle = -r^2 \big\}.$$

It is well known that $\mathbb{S}_1^3(r)$ and $\mathbb{H}_1^3(-r)$ are Lorentzian totally umbilical hypersurfaces with constant sectional curvature $+1/r^2$ and $-1/r^2$, respectively. In order to simplify our notation and computations, we will denote by $\mathbb{M}_c^3$ the De Sitter space $\mathbb{S}_1^3 \equiv \mathbb{S}_1^3(1)$ or the anti De Sitter space $\mathbb{H}_1^3 \equiv \mathbb{H}_1^3(-1)$ according to $c = 1$ or $c = -1$, respectively. We will use $\mathbb{R}_q^4$ to denote the corresponding pseudo-Euclidean space where $\mathbb{M}_c^3$ lives, so its metric is given by

$$\langle \cdot, \cdot \rangle = -dx_1^2 + c\, dx_2^2 + dx_3^2 + dx_4^2,$$

and we can write

$$\mathbb{M}_c^3 = \big\{ x \in \mathbb{R}_q^4 \ : \ -x_1^2 + cx_2^2 + x_3^2 + x_4^2 = c \big\}.$$

Let $\psi : M_s^2 \longrightarrow \mathbb{M}_c^3 \subset \mathbb{R}_q^4$ be an isometric immersion of a surface M_s^2 into $\mathbb{M}_c^3$, and let N be a unit vector field normal to M_s^2 in $\mathbb{M}_c^3$, where $\langle N, N \rangle = \varepsilon = \pm 1$. Here $s = 0$ (resp. $s = 1$) if the induced metric on the surface is Riemannian (resp. Lorentzian). Let ∇^0,

$\overline{\nabla}$ and ∇ denote the Levi-Civita connections on $\mathbb{R}^4_q$, $\mathbb{M}^3_c$ and M^2_s, respectively. Then the Gauss and Weingarten formulas are given by

$$\nabla^0_X Y = \nabla_X Y + \varepsilon \langle SX, Y\rangle N - c\langle X, Y\rangle \psi, \tag{1}$$

and

$$SX = -\overline{\nabla}_X N = -\nabla^0_X N,$$

for all tangent vector fields $X, Y \in \mathfrak{X}(M^2_s)$, where $S : \mathfrak{X}(M^2_s) \longrightarrow \mathfrak{X}(M^2_s)$ stands for the shape operator (or Weingarten endomorphism) of M^2_s, with respect to the chosen orientation N. If a vector field X everywhere different from zero satisfies the condition $SX = \lambda X$, for a differentiable function λ, we will say that X is a principal direction of M^2_s with associated principal curvature λ.

It is well-known (see, for instance, [13, pp. 261–262]) that the shape operator S of the surface M^2_s can be expressed, in an appropriate frame, in one of the following types:

$$\text{I. } S \approx \begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{bmatrix}; \qquad \text{II. } S \approx \begin{bmatrix} \kappa & -b \\ b & \kappa \end{bmatrix}, \quad b \neq 0; \qquad \text{III. } S \approx \begin{bmatrix} \kappa & 0 \\ 1 & \kappa \end{bmatrix}. \tag{2}$$

In cases I and II, S is represented with respect to an orthonormal frame, whereas in case III the frame is pseudo-orthonormal. The characteristic polynomial $Q_S(t)$ of the shape operator S is defined by $Q_S(t) = \det(tI - S) = t^2 + a_1 t + a_2$, where the coefficients a_i are given by

$$a_1 = -\mathrm{tr}(S), \qquad a_2 = \det(S). \tag{3}$$

The mean curvature H of M^2_s in $\mathbb{M}^3_c$ is given by $H = (\varepsilon/2)\mathrm{tr}(S)$, and the coefficient a_2 is also called the *second mean curvature or mean curvature of order* 2, H_2. The Newton transformation of M^2_s is the operator $P_1 : \mathfrak{X}(M^2_s) \longrightarrow \mathfrak{X}(M^2_s)$ defined by

$$P_1 = -2\varepsilon H I + S, \tag{4}$$

and by the Cayley-Hamilton theorem we have $S \circ P_1 = -H_2 I$.

The divergence of a vector field X is the differentiable function defined as the trace of the operator ∇X, where $\nabla X(Y) := \nabla_Y X$. Analogously, the divergence of an operator $T : \mathfrak{X}(M^2_s) \longrightarrow \mathfrak{X}(M^2_s)$ is the vector field $\mathrm{div}(T) \in \mathfrak{X}(M^2_s)$ defined as the trace of ∇T, where $\nabla T(X, Y) = (\nabla_X T)Y$.

We have the following properties of P_1 (see [8, 9]).

Lemma 1. *The Newton transformation P_1 satisfies:*
(a) P_1 is self-adjoint and commutes with S.
(b) $\mathrm{tr}(P_1) = -2\varepsilon H$.
(c) $\mathrm{tr}(S \circ P_1) = -2H_2$.
(d) $\mathrm{tr}(S^2 \circ P_1) = -2\varepsilon H H_2$.
(e) $\mathrm{tr}(\nabla_X S \circ P_1) = -\langle \nabla H_2, X\rangle$.
(f) $\mathrm{div}(P_1) = 0$.

Bearing this lemma in mind we obtain

$$\mathrm{div}(P_1(\nabla f)) = \mathrm{tr}\big(P_1 \circ \nabla^2 f\big),$$

where $\nabla^2 f : \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ denotes the self-adjoint linear operator metrically equivalent to the Hessian of f, given by $\langle \nabla^2 f(X), Y\rangle = \langle \nabla_X(\nabla f), Y\rangle$, for all $X, Y \in \mathfrak{X}(M^2_s)$. Associated to the Newton transformation P_1, we can define the second-order linear differential operator $L_1 : \mathcal{C}^\infty(M^2_s) \longrightarrow \mathcal{C}^\infty(M^2_s)$ given by

$$L_1(f) = \mathrm{tr}\big(P_1 \circ \nabla^2 f\big). \tag{5}$$

An interesting property of L_1 is the following

$$L_1(fg) = gL_1(f) + fL_1(g) + 2\langle P_1(\nabla f), \nabla g\rangle, \tag{6}$$

for every couple of differentiable functions $f, g \in C^\infty(M_s^2)$.

Now, we are going to compute L_1 acting on the coordinate components of the immersion ψ, that is, a function given by $\langle e, \psi\rangle$, where $e \in \mathbb{R}_q^4$ is an arbitrary fixed vector.

A direct computation shows that

$$\nabla\langle e, \psi\rangle = e^\top = e - \varepsilon\langle e, N\rangle N - c\langle e, \psi\rangle\psi, \tag{7}$$

where $e^\top \in \mathfrak{X}(M_s^2)$ denotes the tangential component of e. Taking covariant derivative in (7), and using that $\nabla_X^0 e = 0$, jointly with the Gauss and Weingarten formulae, we obtain

$$\nabla_X\nabla\langle e, \psi\rangle = \varepsilon\langle e, N\rangle SX - c\langle e, \psi\rangle X, \tag{8}$$

for every tangent vector field $X \in \mathfrak{X}(M_s^2)$. Finally, by using (5) and Lemma 1, we find that

$$L_1\langle e, \psi\rangle = \varepsilon\langle e, N\rangle \operatorname{tr}(S\circ P_1) - c\langle e, \psi\rangle \operatorname{tr}(P_1) = -2\varepsilon H_2\langle e, N\rangle + 2\varepsilon cH\langle e, \psi\rangle, \tag{9}$$

and then we get

$$L_1\psi = -2\varepsilon H_2 N + 2\varepsilon cH\psi. \tag{10}$$

We will now compute L_1 acting on a function given by $\langle e, N\rangle$. A straightforward computation yields

$$\nabla\langle e, N\rangle = -Se^\top, \tag{11}$$

that jointly with the Weingarten formula leads to

$$\nabla_X\nabla\langle e, N\rangle = -(\nabla_{e^\top}S)X - \varepsilon\langle e, N\rangle S^2X + c\langle e, \psi\rangle SX,$$

for every tangent vector field X. This equation, jointly with Lemma 1 and (5), yields

$$L_1\langle e, N\rangle = \left\langle \nabla H_2, e^\top\right\rangle + 2HH_2\langle e, N\rangle - 2cH_2\langle e, \psi\rangle, \tag{12}$$

and then

$$L_1 N = \nabla H_2 + 2HH_2 N - 2cH_2\psi. \tag{13}$$

On the other hand, equations (6) and (9) lead to

$$\begin{aligned}\varepsilon L_1(L_1\langle e, \psi\rangle) = {} & -2H_2L_1\langle e, N\rangle - 2L_1(H_2)\langle e, N\rangle - 4\big\langle P_1(\nabla H_2), \nabla\langle e, N\rangle\big\rangle \\ & + 2cHL_1\langle e, \psi\rangle + 2cL_1(H)\langle e, \psi\rangle + 4c\big\langle P_1(\nabla H), \nabla\langle e, \psi\rangle\big\rangle.\end{aligned}$$

From here, and using (9), (7) and (12), we get

$$\begin{aligned}\varepsilon L_1\big(L_1\langle e, \psi\rangle\big) = {} & -2H_2\big\langle\nabla H_2, e\big\rangle + 4\big\langle(S\circ P_1)(\nabla H_2), e\big\rangle + 4c\big\langle P_1(\nabla H), e\big\rangle \\ & + \big[-4\varepsilon HH_2(c + \varepsilon H_2) - 2L_1(H_2)\big]\big\langle e, N\big\rangle \\ & + \big[4cH_2^2 + 4\varepsilon H^2 + 2cL_1(H)\big]\langle e, \psi\rangle.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\varepsilon L_1^2\psi = {} & \big[4cP_1(\nabla H) - 3\nabla H_2^2\big] + \big[-4\varepsilon HH_2(c + \varepsilon H_2) - 2L_1(H_2)\big]N \\ & + \big[4cH_2^2 + 4\varepsilon H^2 + 2cL_1(H)\big]\psi.\end{aligned} \tag{14}$$

Equations characterizing the L_1-2-type surfaces. Let us suppose that M_s^2 is a L_1-2-type surface in $\mathbb{R}_q^4$, that is, the position vector field ψ of M_s^2 in $\mathbb{R}_q^4$ can be written as follows

$$\psi = a + \psi_1 + \psi_2, \quad L_1\psi_1 = \lambda_1\psi_1, \quad L_1\psi_2 = \lambda_2\psi_2, \quad \lambda_1 \neq \lambda_2,\ \lambda_i \in \mathbb{R},$$

where a is a constant vector in $\mathbb{R}_q^4$, and ψ_1, ψ_2 are $\mathbb{R}_q^4$-valued non-constant differentiable functions on M_s^2. Since $L_1\psi = \lambda_1\psi_1 + \lambda_2\psi_2$ and $L_1^2\psi = \lambda_1^2\psi_1 + \lambda_2^2\psi_2$, an easy computation shows that

$$L_1^2\psi = (\lambda_1 + \lambda_2)L_1\psi - \lambda_1\lambda_2(\psi - a). \tag{15}$$

From (10) and (15) we obtain

$$\begin{aligned} L_1^2\psi =& \lambda_1\lambda_2 a^\top + [-2\varepsilon(\lambda_1+\lambda_2)H_2 + \varepsilon\lambda_1\lambda_2 \langle a, N\rangle]N \\ &+ [2\varepsilon c(\lambda_1+\lambda_2)H - \lambda_1\lambda_2 + c\lambda_1\lambda_2\langle a,\psi\rangle]\psi. \end{aligned}$$

This equation, jointly with (14), yields the following equations, that characterize the L_1-2-type surfaces in $\mathbb{M}_c^3$:

$$\lambda_1\lambda_2 a^\top = -3\varepsilon\nabla H_2^2 + 4\varepsilon c P_1(\nabla H), \tag{16}$$

$$\lambda_1\lambda_2\langle a, N\rangle = 2(\lambda_1+\lambda_2)H_2 - 4\varepsilon H H_2(c+\varepsilon H_2) - 2L_1(H_2), \tag{17}$$

$$\lambda_1\lambda_2\langle a,\psi\rangle = 4\varepsilon H_2^2 + 4cH^2 - 2\varepsilon(\lambda_1+\lambda_2)H + c\lambda_1\lambda_2 + 2\varepsilon L_1(H). \tag{18}$$

3. Some Examples

In this section we will show examples not only of L_1-2-type surfaces into $\mathbb{M}_c^3$ but also some surfaces that will be useful later in order to give the classification results.

Example 1. (*Totally umbilical surfaces in* $\mathbb{M}_c^3$) Let us begin by showing that the totally umbilical surfaces in $\mathbb{M}_c^3$ are of L_1-1-type. As is well known, the totally umbilical surfaces in $\mathbb{M}_c^3$ are obtained as the intersection of $\mathbb{M}_c^3$ with a hyperplane of $\mathbb{R}_q^4$, and the causal character of the hyperplane determines the type of the surface. More precisely, let $a \in \mathbb{R}_q^4$ be a non-zero constant vector with $\langle a,a\rangle \in \{-1,0,1\}$, and we take the differentiable function $f_a : \mathbb{M}_c^3 \to \mathbb{R}$ defined by $f_a(x) = \langle a, x\rangle$. It is not difficult to see that for every $\tau \in \mathbb{R}$, with $\langle a,a\rangle - c\tau^2 \neq 0$, the set $M(\tau) = f_a^{-1}(\tau) = \{x \in \mathbb{M}_c^3 \mid \langle a,x\rangle = \tau\}$ is a totally umbilical surface in $\mathbb{M}_c^3$. Its Gauss map N and shape operator S are given by $N(x) = (1/\delta)(a - c\tau x)$ and $SX = (c\tau/\delta)X$, where $\delta = \sqrt{|\langle a,a\rangle - c\tau^2|}$. Now, it is easy to obtain $H = \varepsilon c\tau/\delta$ and $H_2 = \tau^2/\delta^2$, where $\varepsilon = \langle N, N\rangle$. From here we get that $M(\tau)$ has constant Gaussian curvature

$$K = c + \varepsilon H_2 = \frac{c\langle a,a\rangle}{\langle a,a\rangle - c\tau^2},$$

and $M(\tau)$ is a Riemannian or Lorentzian surface according to $\langle a,a\rangle - c\tau^2$ is negative or positive, respectively. Now we will analyze the distinct possibilities.

- Case $c = 1$. Then $M(\tau) \subset \mathbb{M}_c^3 = \mathbb{S}_1^3 \subset \mathbb{R}_1^4$ and we have:
  - i) If $\langle a,a\rangle = -1$, then $K = 1/(\tau^2+1)$, $\varepsilon = -1$, and $M(\tau)$ is isometric to a round sphere of radius $\sqrt{\tau^2+1}$, $M(\tau) \equiv \mathbb{S}^2(\sqrt{\tau^2+1})$.
  - ii) If $\langle a,a\rangle = 0$, then $\tau \neq 0$, $K = 0$, $\varepsilon = -1$, and $M(\tau)$ is isometric to the Euclidean plane, $M(\tau) \equiv \mathbb{R}^2$. Then $M(\tau)$ is a flat totally umbilic surface.
  - iii) If $\langle a,a\rangle = 1$, then either $|\tau| > 1$, $K = -1/(\tau^2-1)$, $\varepsilon = -1$, and $M(\tau)$ is isometric to the hyperbolic plane of radius $-\sqrt{\tau^2-1}$, $M(\tau) \equiv \mathbb{H}^2(-\sqrt{\tau^2-1})$, or $|\tau| < 1$,

$K = 1/(1-\tau^2)$, $\varepsilon = 1$, and $M(\tau)$ is isometric to the De Sitter space of radius $\sqrt{1-\tau^2}$, $M(\tau) \equiv \mathbb{S}^2_1(\sqrt{1-\tau^2})$.

- Case $c = -1$. Then $M(\tau) \subset \mathbb{M}^3_c = \mathbb{H}^3_1 \subset \mathbb{R}^4_2$ and we have:
  i) If $\langle a, a\rangle = -1$, then either $|\tau| > 1$, $K = 1/(\tau^2-1)$, $\varepsilon = 1$, and $M(\tau)$ is isometric to a De Sitter space of radius $\sqrt{\tau^2-1}$, $M(\tau) \equiv \mathbb{S}^2_1(\sqrt{\tau^2-1})$, or $|\tau| < 1$, $K = -1/(1-\tau^2)$, $\varepsilon = -1$, and M_τ is isometric to a hyperbolic plane of radius $-\sqrt{1-\tau^2}$, $M(\tau) \equiv \mathbb{H}^2(-\sqrt{1-\tau^2})$.
  ii) If $\langle a, a\rangle = 0$, then $\tau \neq 0$, $K = 0$, $\varepsilon = 1$, and $M(\tau)$ is isometric to the Lorentz-Minkowski plane, $M(\tau) \equiv \mathbb{R}^2_1$. Then $M(\tau)$ is a flat totally umbilical surface.
  iii) If $\langle a, a\rangle = 1$, then $K = -1/(\tau^2+1)$, $\varepsilon = 1$, and $M(\tau)$ is isometric to the Lorentzian hyperbolic plane, $M(\tau) \equiv \mathbb{H}^2_1(-\sqrt{\tau^2+1})$.

Bearing (10) in mind, we obtain

$$L_1\psi = \lambda\psi + b, \quad \text{where } \lambda = \frac{2\tau}{\delta^3}\left(\varepsilon c\tau^2 + \delta^2\right) \text{ and } b = -\frac{2\varepsilon\tau^2}{\delta^3}a.$$

We distinguish three cases:

- If $\tau = 0$ (and so $S = 0$), then $L_1\psi = 0$ and $M(0)$ is a null L_1-1-type surface (for $c = 1$, $M(0) \equiv \mathbb{S}^2$ or $M(0) \equiv \mathbb{S}^2_1$, and for $c = -1$, $M(0) \equiv \mathbb{H}^2$ or $M(0) \equiv \mathbb{H}^2_1$).
- If $\varepsilon c\tau^2 + \delta^2 = 0$, then $L_1\psi = b \neq 0$ and so $M(\tau)$ is of infinite L_1-type. Observe that if $c = 1$ then $M(\tau) \equiv \mathbb{R}^2$, and if $c = -1$ then $M(\tau) \equiv \mathbb{R}^2_1$. In any case, $M(\tau)$ is a flat totally umbilical surface.
- If $\lambda \neq 0$, then we can write $\psi = \psi_0 + \psi_1$ with $\psi_0 = -(1/\lambda)b$ and $\psi_1 = \psi + (1/\lambda)b$. Then $L_1\psi_1 = \lambda\psi_1$ showing that $M(\tau)$ is of L_1-1-type.

Example 2. (*Standard pseudo-Riemannian products in* $\mathbb{M}^3_c$) In this example we show, as in the Riemannian case, that the pseudo-Riemannian products are L_1-2-type surfaces. Let $f : \mathbb{M}^3_c \longrightarrow \mathbb{R}$ be the differentiable function defined by

$$f(x) = cx_1^2 - \delta_2 x_2^2 + \delta_3 x_3^2 + \delta_4 x_4^2,$$

where $\delta_2, \delta_3, \delta_4 \in \{0, 1\}$ with $\delta_2 + \delta_3 + \delta_4 = 1$. In short, $f(x) = \langle Dx, x\rangle$, where D is the diagonal matrix $D = \mathrm{diag}[c, \delta_2, \delta_3, \delta_4]$. Then, for every $r > 0$, and $\rho = \pm 1$ with $r^2 - c\rho \neq 0$, the level set $M^2 = f^{-1}(\rho r^2)$ is a surface in $\mathbb{M}^3_c$, unless it is empty.

The Gauss map and the shape operator are given by

$$N(x) = \frac{1}{r\sqrt{|\rho - cr^2|}}\left(Dx - \rho cr^2 x\right) \quad \text{and} \quad S = \frac{-1}{r\sqrt{|\rho - cr^2|}}\begin{bmatrix} 1-\rho cr^2 & \\ & -\rho cr^2 \end{bmatrix}. \tag{19}$$

Therefore, by using (10) we get that $L_1\psi = \lambda_1 D\psi + \lambda_2(\psi - D\psi)$, where

$$\lambda_1 = \frac{2\varepsilon H_2(\rho cr^2 - 1)}{r\sqrt{|\rho - cr^2|}} + 2\varepsilon cH \qquad \text{and} \qquad \lambda_2 = \frac{2\varepsilon H_2 \rho cr^2}{r\sqrt{|\rho - cr^2|}} + 2\varepsilon cH.$$

If we put $\psi_1 = D\psi$ and $\psi_2 = \psi - D\psi$, then $\psi = \psi_1 + \psi_2$, $L_1\psi_1 = \lambda_1\psi_1$, and $L_1\psi_2 = \lambda_2\psi_2$. Therefore, M^2 is an L_1-2-type surface in $\mathbb{R}^4_q$.

The following two tables show the distinct pseudo Riemannian products in $\mathbb{M}^3_c$.

Standard products in $\mathbb{S}^3_1$				
δ_2	δ_3	δ_4	ρ	surface
1	0	0	-1	$\mathbb{H}^1(-r) \times \mathbb{S}^1(\sqrt{1+r^2})$
1	0	0	1	$\mathbb{S}^1_1(r) \times \mathbb{S}^1(\sqrt{1-r^2})$
0	1	0	1	$\mathbb{S}^1(r) \times \mathbb{S}^1_1(\sqrt{1-r^2})$ or $\mathbb{S}^1(r) \times \mathbb{H}^1(-\sqrt{r^2-1})$
0	0	1	1	

Standard products in $\mathbb{H}^3_1$				
δ_2	δ_3	δ_4	ρ	surface
1	0	0	-1	$\mathbb{H}^1_1(-r) \times \mathbb{S}^1(\sqrt{r^2-1})$
0	1	0	1	$\mathbb{S}^1_1(r) \times \mathbb{H}^1(-\sqrt{1+r^2})$
0	0	1	1	
0	1	0	-1	$\mathbb{H}^1(-r) \times \mathbb{S}^1(\sqrt{r^2-1})$ or $\mathbb{H}^1(-r) \times \mathbb{H}^1(-\sqrt{1-r^2})$
0	0	1	-1	

Example 3. (*Complex circle*) This example shows a surface satisfying an equation similar to (15) but it is not a surface of L_1-2-type. Given a complex number $k = a + bi$, where $a, b \in \mathbb{R}$ such that $a^2 - b^2 = -1$ and $ab \neq 0$, the complex circle of radius k is defined by Magid in [12] as follows. Since $\mathbb{C}^2$ can be identified with $\mathbb{R}^4_2$ by sending $(z, w) = (u_1 + iu_2, x + iy)$ to (u_1, x, u_2, y), the mapping $\psi : \mathbb{C} \equiv \mathbb{R}^2_1 \to \mathbb{H}^3_1 \subset \mathbb{C}^2 \equiv \mathbb{R}^4_2$ given by $\psi(z) = k(\cos z, \sin z)$ is an isometric immersion of $\mathbb{R}^2_1$ into $\mathbb{R}^4_2$ with parallel second fundamental form. ψ parameterizes a Lorentzian surface in $\mathbb{H}^3_1$ known as the *complex circle* of radius $k = a + bi$, and the corresponding shape operator S is expressed as

$$\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}, \quad \text{with } \alpha = \frac{2ab}{a^2+b^2} \quad \text{and} \quad \beta = \frac{1}{a^2+b^2}.$$

Thus, a complex circle is a flat surface in $\mathbb{H}^3_1$ with constant mean curvature $H = \alpha$. Finally, a straightforward computation yields

$$L_1^2\psi = \frac{-4}{(a^2+b^2)^2}\psi,$$

showing that the complex circle satisfies an equation similar to (15). However, it is not an L_1-2-type surface since there are no two distinct real numbers λ_1, λ_2 satisfying (15).

Example 4. (*B-scroll*) Finally, we will show an example with non-diagonalizable shape operator. This example also shows the importance of the Gaussian curvature in concluding whether or not the surface is of L_1-2-type. Let $\gamma(s)$, $s \in I$, be a null curve in $\mathbb{M}^3_c \subset \mathbb{R}^4_q$ with associated Cartan frame $\{A = \gamma', B, C\}$, i.e., $\{A, B, C\}$ is a pseudo-orthonormal frame of vector fields along $\gamma(s)$

$$\langle A, A\rangle = \langle B, B\rangle = 0, \langle A, B\rangle = -1, \langle A, C\rangle = \langle B, C\rangle = 0, \langle C, C\rangle = 1,$$

such that

$$\gamma'(s) = A(s), \quad C'(s) = -a_0A(s) - \kappa(s)B(s),$$

where a_0 is a nonzero constant and $\kappa(s) \neq 0$ for all s. Then the map $\psi : I \times \mathbb{R} \to \mathbb{M}^3_c \subset \mathbb{R}^4_q$, given by $\psi(s,u) = \gamma(s) + uB(s)$, defines a Lorentzian surface M^2_1 in $\mathbb{M}^3_c$ known as a B-scroll on the null curve γ (see [6] and [7]).

A simple calculation shows that the vector field $N(s,u)$ given by $N(s,u) = -a_0 uB(s) + C(s)$ defines a unit normal vector field to M^2_1 in $\mathbb{M}^3_c$, and the shape operator S is expressed in the usual tangent frame $\{\partial_s\psi, \partial_u\psi\}$ as

$$S = \begin{bmatrix} a_0 & 0 \\ \kappa(s) & a_0 \end{bmatrix},$$

whose minimal polynomial is $(t - a_0)^2$. Consequently, a B-scroll is a Lorentzian surface with constant mean curvature $H = a_0$ and constant Gauss curvature $K = c + a_0^2$.

Now, let us check if the B-scroll is of L_1-2-type. From (14) and (10) we find

$$L_1^2\psi = (2HK)L_1\psi.$$

Therefore, a non-flat B-scroll is a null L_1-2-type surface, whereas a flat B-scroll (i.e, $c = -1$, $H_2 = a_0^2 = 1$) is an L_1-biharmonic surface of infinite type.

4. Main results

Now, we are in position to present our first main result.

Theorem 2. *Let $\psi : M^2_s \to \mathbb{M}^3_c \subset \mathbb{R}^4_q$ be an orientable surface of L_1-2-type. Then H is constant if and only if H_2 is constant.*

Proof. Let us suppose that H is constant. Our goal is to prove that H_2 is also constant. Otherwise, let us consider the non-empty open set

$$\mathcal{U}_2 = \{p \in M^2_s \ : \ \nabla H_2^2(p) \neq 0\}.$$

By taking covariant derivative in (18) we have $\lambda_1\lambda_2 a^\top = 4\varepsilon\nabla H_2^2$. From here and (16) we get $\nabla H_2^2 = 0$ on $\mathcal{U}_2$, which is a contradiction. Hence, H_2 is constant.

Conversely, let us suppose now that H_2 is constant, and consider the open set

$$\mathcal{U}_1 = \{p \in M^2_s \ : \ \nabla H^2(p) \neq 0\}.$$

Our goal is to show that $\mathcal{U}_1$ is empty. Otherwise, by taking covariant derivative in (17) we get

$$\lambda_1\lambda_2 S a^\top = 4\varepsilon H_2(c + \varepsilon H_2)\nabla H.$$

From (16), and bearing in mind that $S \circ P_1 = -H_2 I$, we get $\lambda_1\lambda_2 S a^\top = -4\varepsilon c H_2 \nabla H$, so that

$$4\varepsilon H_2(2c + \varepsilon H_2)\nabla H = 0.$$

Consequently, on $\mathcal{U}_1$ we have either $H_2 \equiv -2\varepsilon c$ or $H_2 \equiv 0$. We will study each case separately.

Case 1: $H_2 \equiv -2\varepsilon c$. By applying the operator L_1 on both sides of (17), and using (18), we get

$$\lambda_1\lambda_2 L_1 \langle a, N\rangle = -8L_1(H) = -4\big[\varepsilon\lambda_1\lambda_2 \langle a, \psi\rangle - 4\varepsilon c H^2 + 2(\lambda_1 + \lambda_2)H - \varepsilon c\lambda_1\lambda_2 - 16\big].$$

By using (12) we obtain

$$-c\lambda_1\lambda_2 \langle a, N\rangle H + \lambda_1\lambda_2 \langle a, \psi\rangle = -\lambda_1\lambda_2 \langle a, \psi\rangle + 4cH^2 - 2\varepsilon(\lambda_1 + \lambda_2)H + c\lambda_1\lambda_2 + 16\varepsilon,$$

and from (17) we find

$$\lambda_1\lambda_2\langle a,\psi\rangle = -2cH^2 - 3\varepsilon(\lambda_1+\lambda_2)H + \frac{c\lambda_1\lambda_2}{2} + 8\varepsilon.$$

By taking covariant derivative here, and using (16), we get

$$\big[-4cH - 3\varepsilon(\lambda_1+\lambda_2)\big]\nabla H = -8cH\nabla H + 4\varepsilon c S(\nabla H),$$

Now, by applying S on both sides of this equation, and taking in mind that $S\circ P_1 = 2\varepsilon c I$, we have

$$\big[-4cH - 3\varepsilon(\lambda_1+\lambda_2)\big]S(\nabla H) = 4\varepsilon c S\circ P_1(\nabla H) = 8\nabla H.$$

From the last two equations we deduce

$$16H^2 - 9(\lambda_1+\lambda_2)^2 + 32\varepsilon c = 0,$$

and then H is constant on $\mathcal{U}_1$, which is a contradiction.

Case 2: $H_2 \equiv 0$, then $S(S-2\varepsilon HI) = 0$. Since a totally umbilical surface is not of L_1-2-type, we can assume that M_s^2 has two distinct principal curvatures $\kappa_1 = 0$ and $\kappa_2 = 2\varepsilon H \neq 0$. Let $\{E_1, E_2\}$, with $\langle E_i, E_i\rangle = \varepsilon_i$, be a local orthonormal frame of principal directions of S such that

$$S = \begin{pmatrix} 0 & 0 \\ 0 & 2\varepsilon H \end{pmatrix}.$$

Write $\nabla H = \alpha E_1 + \beta E_2$, where $\alpha = \varepsilon_1 E_1(H)$, then we have $S(\nabla H) = 2\varepsilon H\beta E_2$. On the other hand, the Codazzi equation $(\nabla_{E_1}S)E_2 = (\nabla_{E_2}S)E_1$ yields

$$\varepsilon_2 E_1(H) = -H\langle \nabla_{E_2}E_1, E_2\rangle \quad \text{and} \quad 0 = \langle \nabla_{E_1}E_2, E_1\rangle.$$

From here we deduce

$$\begin{aligned} &\nabla_{E_1}E_1 = 0, && \nabla_{E_1}E_2 = 0, \\ &\nabla_{E_2}E_1 = -\frac{\varepsilon_1\alpha}{H}E_2, && \nabla_{E_2}E_2 = \frac{\varepsilon_1\varepsilon_2\alpha}{H}E_1, \quad [E_1,E_2] = \frac{\varepsilon_1\alpha}{H}E_2, \end{aligned}$$

and then the curvature tensor is given by ([13])

$$R(E_1,E_2)E_1 = \nabla_{E_1}\nabla_{E_2}E_1 - \nabla_{E_2}\nabla_{E_1}E_1 - \nabla_{[E_1,E_2]}E_1 = \Big[-E_1\Big(\frac{\varepsilon_1\alpha}{H}\Big) + \Big(\frac{\varepsilon_1\alpha}{H}\Big)^2\Big]E_2.$$

On the other hand, since $\mathbb{M}_c^3$ is of constant curvature we have

$$R(E_1,E_2)E_1 = -\varepsilon_1 c E_2.$$

From the last two equations we have

$$HE_1(\alpha) = cH^2 + 2\varepsilon_1\alpha^2. \tag{20}$$

Now, since $P_1(E_1) = -\kappa_2 E_1$ and $P_1(E_2) = 0$, we get

$$L_1(H) = -2\varepsilon H E_1(\alpha), \tag{21}$$

that jointly with (18) yields

$$\lambda_1\lambda_2\langle a,\psi\rangle = -2\varepsilon(\lambda_1+\lambda_2)H + c\lambda_1\lambda_2 - 8\varepsilon_1\alpha^2,$$

and from here we get

$$E_1(\lambda_1\lambda_2\langle a,\psi\rangle) = -2\varepsilon(\lambda_1+\lambda_2)\varepsilon_1\alpha - 16\varepsilon_1\alpha E_1(\alpha). \tag{22}$$

However, from (16) we have

$$\lambda_1\lambda_2 a^\top = -8cH\alpha E_1, \tag{23}$$

so that

$$E_1(\lambda_1\lambda_2 \langle a, \psi\rangle) = \left\langle \lambda_1\lambda_2 a^\top, E_1\right\rangle = -8\varepsilon_1 cH\alpha, \tag{24}$$

that jointly with (22) implies

$$-4E_1(\alpha) = -2cH + \frac{\varepsilon(\lambda_1+\lambda_2)}{2}. \tag{25}$$

From here, and using (21) and (18), we find

$$\lambda_1\lambda_2 \langle a, \psi\rangle = 2cH^2 - \frac{3\varepsilon}{2}(\lambda_1+\lambda_2)H + c\lambda_1\lambda_2. \tag{26}$$

Taking gradient in (26) and using (16) we get

$$\Big[4cH - \frac{3\varepsilon}{2}(\lambda_1+\lambda_2)\Big]\nabla H = 4\varepsilon c P_1(\nabla H) = -8cH\nabla H + 4\varepsilon c S(\nabla H). \tag{27}$$

Now, by applying S on both sides of (27) we have

$$\big[4cH - \frac{3\varepsilon}{2}(\lambda_1+\lambda_2)\big] S(\nabla H) = 0.$$

From the last two equations we find

$$\big[4cH - \frac{3\varepsilon}{2}(\lambda_1+\lambda_2)\big]\Big[3\varepsilon H - \frac{3c}{8}(\lambda_1+\lambda_2)\Big]\nabla H = 0,$$

and this implies that H is constant on $\mathcal{U}_1$, which is a contradiction. This finishes the proof of Theorem 2. □

Theorem 3. *Let $\psi : M_s^2 \to \mathbb{M}_c^3 \subset \mathbb{R}_q^4$ be an orientable surface of L_1-2-type. The following conditions are equivalent to each other:*

1) *M_s^2 has a constant principal curvature.*
2) *H is constant.*
3) *H_2 is constant.*

Proof. From Theorem 2 we know that conditions 2) and 3) are equivalent to each other. Therefore, it is sufficient to prove the equivalence of conditions 1) and 2).

If condition 2) holds then M_s^2 is an isoparametric surface since condition 3) also holds. Then the shape operator S of M_s^2 in $\mathbb{M}_c^3$ is of type I (with constant κ_1 and κ_2) or type III (with constant κ), so we have condition 1). Observe that S cannot be of type II since this kind of surfaces do not exist in $\mathbb{S}_1^3$ and the complex circle in $\mathbb{H}_1^3$ is not of L_1-2-type.

Let us now assume that condition 1) is satisfied, then S is of type I or type III. If S is of type III then κ is constant, and therefore H and H_2 are also constants.

Let us consider the case where S is diagonalizable with two distinct principal curvatures $\kappa_1 \neq \kappa_2$, and assume without loss of generality that κ_1 is a nonzero constant (otherwise, $H_2 = 0$ and the Theorem 2 applies). Consider the open set

$$\mathcal{U} = \big\{p \in M_s^2 \ : \ \nabla\kappa_2^2(p) \neq 0\big\}.$$

Our goal is to show that $\mathcal{U}$ is empty. Otherwise, the equations (16)–(18) of a L_1-2-type surface can be rewritten in terms of κ_2 as follows

$$\lambda_1\lambda_2 a^\top = [-6\varepsilon\kappa_1^2\kappa_2 - 2c(\kappa_1+\kappa_2)]\nabla\kappa_2 + 2cS(\nabla\kappa_2), \tag{28}$$

$$\lambda_1\lambda_2 \langle a, N\rangle = 2\kappa_1\kappa_2[\lambda_1+\lambda_2 - (\kappa_1+\kappa_2)(c+\varepsilon\kappa_1\kappa_2)] - 2\kappa_1 L_1\kappa_2, \tag{29}$$

$$\lambda_1\lambda_2 \langle a, \psi\rangle = 4\varepsilon\kappa_1^2\kappa_2^2 + c(\kappa_1+\kappa_2)^2 - (\lambda_1+\lambda_2)(\kappa_1+\kappa_2) + c\lambda_1\lambda_2 + L_1\kappa_2. \tag{30}$$

From (29) and (30) we find

$$\lambda_1\lambda_2 \langle a, N\rangle = -2\kappa_1\lambda_1\lambda_2 \langle a, \psi\rangle + 2\kappa_1[3\varepsilon\kappa_1^2\kappa_2^2 + c\kappa_1^2 + c\kappa_1\kappa_2 - (\lambda_1+\lambda_2)\kappa_1 + c\lambda_1\lambda_2 - \varepsilon\kappa_1\kappa_2^3].$$

By applying the gradient here we obtain

$$-\lambda_1\lambda_2 S a^\top = -2\kappa_1\lambda_1\lambda_2 a^\top + 2\kappa_1^2[c + 6\varepsilon\kappa_1\kappa_2 - 3\varepsilon\kappa_2^2]\nabla\kappa_2. \tag{31}$$

On the other hand, by (28) we get

$$\lambda_1\lambda_2 S a^\top = -6\varepsilon\kappa_1^2\kappa_2 S(\nabla\kappa_2) - 2c\kappa_1\kappa_2\nabla\kappa_2 \tag{32}$$

Now, from (28), (31) and (32) we deduce

$$(3\varepsilon\kappa_1\kappa_2 + 2c)S(\nabla\kappa_2) = (12\varepsilon\kappa_1^2\kappa_2 - 3\varepsilon\kappa_1\kappa_2^2 + c\kappa_2 + 3c\kappa_1)\nabla\kappa_2.$$

Since $3\kappa_1\kappa_2 + 2c \neq 0$ (otherwise, κ_2 would be constant), we deduce

$$S(\nabla\kappa_2) = f(\kappa_1,\kappa_2)\nabla\kappa_2, \qquad f(\kappa_1,\kappa_2) = \frac{12\varepsilon\kappa_1^2\kappa_2 - 3\varepsilon\kappa_1\kappa_2^2 + c\kappa_2 + 3c\kappa_1}{(3\varepsilon\kappa_1\kappa_2 + 2c)}.$$

This equation implies that either $f(\kappa_1,\kappa_2) = \kappa_1$ or $f(\kappa_1,\kappa_2) = \kappa_2$. In any case it follows that κ_2 is constant on $\mathcal{U}$, and this is a contradiction. This finishes the proof of Theorem 3. $\square$

As a consequence of theorems 2 and 3, the examples 1–4 and the classification of isoparametric surfaces in $\mathbb{M}_c^3$, see [1, 14, 15], we have the following characterization of L_1-2-type surfaces in $\mathbb{M}_c^3$.

Theorem 4. *Let $\psi : M_s^2 \to \mathbb{S}_1^3 \subset \mathbb{R}_1^4$ be an orientable surface of L_1-2-type. Then one of the following conditions is satisfied*

- *M_s^2 an open portion of a standard pseudo-Riemannian product: $\mathbb{S}_1^1(\sqrt{1-r^2})\times\mathbb{S}^1(r)$ or $\mathbb{H}^1(\sqrt{1-r^2})\times\mathbb{S}^1(r)$.*
- *M_1^2 is a B-scroll over a null curve.*
- *M_s^2 has non constant mean curvature, non constant Gaussian curvature, and non constant principal curvatures.*

Theorem 5. *Let $\psi : M_s^2 \to \mathbb{H}_1^3 \subset \mathbb{R}_2^4$ be an orientable surface of L_1-2-type. Then one of the following conditions is satisfied*

- *M_s^2 is an open portion of a standard pseudo-Riemannian product: $\mathbb{S}_1^1(r)\times\mathbb{H}^1(-\sqrt{r^2+1})$, or $\mathbb{H}^1(-r)\times\mathbb{H}^1(-\sqrt{1-r^2})$, or $\mathbb{H}_1^1(-r)\times\mathbb{S}^1(\sqrt{r^2-1})$*
- *M_1^2 is a non-flat B-scroll over a null curve.*
- *M_s^2 has non constant mean curvature, non constant Gaussian curvature, and non constant principal curvatures.*

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Conflict of interest

The authors declare that they have no conflict of interest.

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